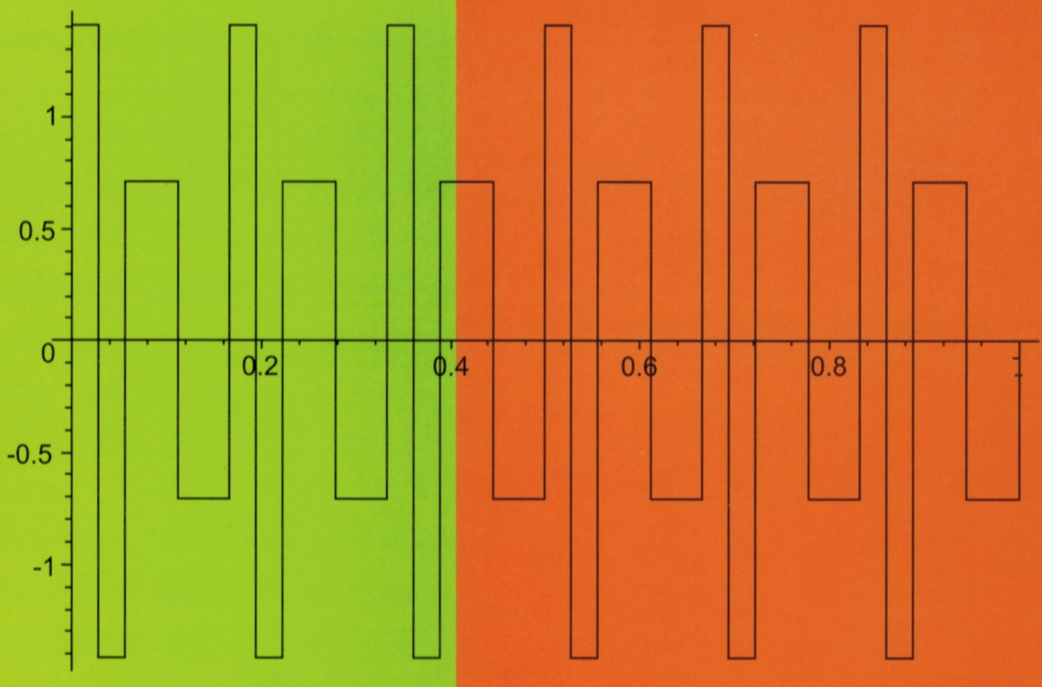
Jorge Bustamante González
Slaviša V. Djordjevic
Miguel A. Jiménez Pozo

TÓPICOS DE TEORÍA DE LA APROXIMACIÓN IV



textos Científicos

Benemérita Universidad Autónoma de Puebla

Topics in Approximation Theory IV

Tópicos de Teoría de la Aproximación IV

Jorge Bustamante González

Slaviša V. Djordjević

Miguel A. Jiménez Pozo

BENEMÉRITA UNIVERSIDAD AUTÓNOMA DE PUEBLA

Vicerrectoría de Investigación y Estudios de Posgrado

Facultad de Ciencias Fìsico Matemáticas

Dirección de Fomento Editorial

Benemérita Universidad Autónoma de Puebla

Enrique Agüera Ibáñez
Rector

José Ramón Equíbar Cuenca
Secretario General

Pedro Hugo Hernández Tejeda
Vicerrector de Investigación y Estudios de Posgrado

María Lilia Cedillo Ramírez
Vicerrectora de Extensióón y Difusión de la Cultura

Cupatitzio Ramírez Romero
Director de la Facultad de Ciencias Fìsico Matemáticas

Carlos Contreras Cruz
Dirección Editorial

Primera edición 2012
ISBN: 978–607–487–467–9

© Benemérita Universidad Autónoma de Puebla
Dirección de Fomento Editorial
2 Norte 1404, CP 72000 Puebla, Pue.
Teléfono y Fax 01 222 246 8559

Impreso y hecho en México
Printed and made in Mexico

Index

1 Preface **9**

2 Self-adjoint Riesz projections for Hilbert space operators **11**
2.1 Introduction 11
2.2 Results 13
2.3 Applications 15
2.4 References 17

3 Upper estimates for one-sided trigonometric approximation **21**
3.1 Introduction 22
3.2 Some properties of the moduli 23
3.3 Main result 28
3.4 References 31

4 Where algebra and topology meet: a cautionary tale **33**
4.1 Algebra 33
4.2 Topology 34
4.3 Neumann series 36
4.4 Algebraic closure 37
4.5 Algebraic foreclosure 38
4.6 Rescue package 38
4.7 Housekeeping 40
4.8 Nearly invertibles 41
4.9 Fredholm theory 42

4.10 Stable range 45
4.11 References 46

5 Approximation on compact totally disconnect groups 49
5.1 Introduction 50
5.2 The Walsh functions 51
5.3 The Dyadic group 52
5.4 Vilenkin systems 56
5.5 Results and open problems 58
5.6 Representative product systems 73
5.7 Convergence in L^p-norm of Fourier series 78
5.8 Relation with the interval $[0, 1[$ 80
5.9 Dirichlet kernels 83
5.10 Fejér means and Cesàro means of order α 85
5.11 References 87

6 Fredholm theory and Kato decomposition 95
6.1 Introduction 95
6.2 The spectral decomposition 98
6.3 The Kato decomposition 103
6.4 Fredholm theory through invertibility 111
6.5 Banach algebras 118
6.6 References 121

7 Generalized uniform weighted approximation of continuous functions from Haar spaces 125
7.1 Introduction 126
7.2 Chebyshev asymmetric approximation by generalized polynomials. 128
7.3 Uniqueness of the best approximation 132
7.4 Extensions and applications 134
7.5 References 138

8 Mejora del desempeño de una lente triplete con el método de relajación **141**
8.1 Introducción: Diseño Automático de lentes mediante la solución de desigualdades. 142
8.2 El método de relajación 144
8.2.1 Algoritmo de Relajación 144
8.3 ¿Qué es Sage? 145
8.4 Codificación del método de relajación en Sage 146
8.5 Resultados 149
8.6 Conclusiones 153
8.7 Referencias 153

9 Stability of Closed-Convex-Valued Mappings: Survey **157**
9.1 Introduction 158
9.2 Closed-convex-valued mappings and the associated boundaries 161
9.2.1 Preliminaries 162
9.2.2 Some exploratory examples 165
9.2.3 Lower semicontinuity 169
9.2.4 Closedness 170
9.2.5 Upper semicontinuity 171
9.2.6 Continuity 171
9.3 The relative boundary and extreme point set 173
9.3.1 Preliminaries 173
9.3.2 Lower semicontinuity 174
9.3.3 Upper semicontinuity and closedness 174
9.4 References 178

10 Some estimates for one-sided trigonometric approximation **181**
10.1 Introduction 181
10.2 Steklov type functions 184
10.3 Estimates for one-sided approximation 187
10.4 References 190

1

Preface

Following the idea of the previous three volumes, the present one contains original research works and surveys presented in the 4^{th} Colloquium of the series on Approximation Theory and Related Topics, Faculty of Physics and Mathematics at Benemérita Universidad Autónoma de Puebla.

The meeting was held during five sessions: 1^{st} in November 05–07, 2009, 2^{nd} in December 02–05, 2009; 3^{th} in January 03–05, 2010, 4^{th} in February 04–05, 2010 and 5^{th} in February 25–26, 2010.

The monographs in the serial are mainly useful for graduate students as well as researchers in other areas. With this aim at hands we collect here several talks of the Colloquium and emphasize that all papers in the volume, the ones of the editors included, have been submitted to usual referee procedure.

We would like to thank the University of Puebla, the Organizing Committee of the Colloquium, their participants, the authors, the anonymous referees and all who make possible the edition of this book.

Jorge Bustamante González
Slaviša V. Djordjević
Miguel A. Jiménez Pozo

2

Self-adjoint Riesz projections for Hilbert space operators

B. P. Duggal[1] and S. V. Djordjević[2]

Abstract: If a Hilbert space operator T is polar of order k for some $k \in \mathbf{N}$ at a point $\lambda \in \mathrm{iso}\sigma(T)$ and $(\lambda - T)^{-k}(0) \subseteq ((\lambda - T)^*)^{-k}(0)$, then the Riesz projection $P_T(\lambda)$ associated with λ is self-adjoint.

AMS Subject Classification: Primary: 47B20, 47A10, 47A11.

Keywords and phrases: Hilbert space, spectrum, isolated point, Riesz projection, self-adjoint.

2.1 Introduction

The Riesz projection , or the spectral projection, $P_T(\lambda)$ associated with an isolated point λ of the spectrum $\sigma(T)$, $\lambda \in \mathrm{iso}\sigma(T)$, of a Banach space

[1]8 Redwood Grove, Northfield Avenue, Ealing, London W5 4SZ,United Kingdom, e-mail: bpduggal@yahoo.co.uk

[2]Faculta de Ciencias Físico Matemáticas, Benemérita Universidad Autónoma de Puebla, e-mail: slavdj@fcfm.buap.mx

operator $T \in B(\mathcal{X})$ is the idempotent

$$P_T(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_\lambda} (\mu - T)^{-1} d\mu,$$

where $\mu - T = \mu I - T$ and Γ_λ is a positively oriented simple closed path with λ, and no other point of $\sigma(T)$, in its interior. We recall, [10] and [13], that the range of $P_T(\lambda)$, $P_T(\lambda)\mathcal{X}$, equals the quasinilpotent part $H_0(\lambda - T) = \{x \in \mathcal{X} : \lim_{n\to\infty} ||(\lambda - T)^n x||^{\frac{1}{n}} = 0\}$ of $\lambda - T$ and the kernel of $P_T(\lambda)$, $P_T(\lambda)^{-1}(0)$, coincides with the analytic core $K(\lambda - T)$,

$$K(\lambda - T) = \{x \in \mathcal{X} : \text{there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0 \text{ for which}$$
$$x = x_0, (\lambda - T)x_{n+1} = x_n \ \text{ and } \ \|x_n\| \leq \delta^n \|x\| \ \text{ for all } \ n = 1, 2, \cdots\},$$

of $\lambda - T$. Here, for $\lambda \in \mathrm{iso}\sigma(T)$, $H_0(\lambda - T)$ and $K(\lambda - T)$ are closed hyperinvariant subspaces of T such that $(\lambda - T)^{-q}(0) \subseteq H_0(\lambda - T)$ for all $q = 0, 1, 2, ...$ and $(\lambda - T)K(\lambda - T) = K(\lambda - T)$.

Let $\mathcal{H}$ denote an infinite dimensional complex Hilbert space, $B(\mathcal{H})$ the algebra of operators on $\mathcal{H}$, and let $T \in B(\mathcal{H})$. The problem of determining points $\lambda \in \mathrm{iso}\sigma(T)$ for which the Riesz projection $P_T(\lambda)$ is self adjoint has been considered by a number of authors in the recent past (see [2], [4], [8], [11], [12], [14], [15], and [18]). Thus, if $T \in B(\mathcal{H})$ is hyponormal, then $P_T(\lambda)$ is self-adjoint for every $\lambda \in \mathrm{iso}\sigma(T)$ [14]; more generally, if $T \in B(\mathcal{H})$ is p-hyponormal $(0 < p \leq 1)$ or M-hyponormal, then $P_T(\lambda)$ is self-adjoint for every $\lambda \in \mathrm{iso}\sigma(T)$ [2], [4]. Again, if $T \in B(\mathcal{H})$ is w-hyponormal or (p, q)-quasihyponormal or quasi-class A or paranormal, then $P_T(\lambda)$ is self-adjoint for every non-zero $\lambda \in \mathrm{iso}\sigma(T)$ ([4], [8], [11], [12], [15] and [16]). (All these classes of operators are defined in the sequel.) In general, if $T \in B(\mathcal{H})$ is *totally hereditarily normaloid* (in the sense of [3]), then the points $\lambda \in \sigma(T)$ are (simple poles of the resolvent, hence) eigen-values of T; if such an eigen-value is normal, i.e. the corresponding eigen-space is reducing, then $P_T(\lambda)$ is self-adjoint [4] (Corollary 2.1). In the following, we generalize this result to prove that "if a point $\lambda \in \mathrm{iso}\sigma(T)$ is a pole (of some order k) of the resolvent of $T \in B(\mathcal{H})$ and if T satisfies the property that $(T - \lambda)^{-k}(0) \subseteq (T - \lambda)^{*-k}(0)$, then the Riesz projection associated with λ is self-adjoint".

Recall [9] (p. 248] that an operator $T \in B(\mathcal{X})$ is polar (resp., simply polar) at $\lambda \in \mathrm{iso}\sigma(T)$ if and only if it has a *Drazin inverse* $S \in B(\mathcal{X})$ for which, with $k \in \mathbf{N}$ (resp., $k = 1$),

$$(\lambda - T)^k S(\lambda - T) - (\lambda - T)^k = 0 = (\lambda - T)S - S(\lambda - T), \;\; S(\lambda - T)S = S:$$

S is unique and double commutes with T, and the spectral projection $P_T(\lambda)$ associated with λ is then given by

$$P_T(\lambda) = I - S(\lambda - T) = I - (\lambda - T)S.$$

In this note we establish when $P_T(\lambda)$ is self adjoint. We say that an operator $A \in B(\mathcal{H})$ has property (N), the normal eigenvalue property, at 0 if there is inclusion

$$A^{-1}(0) \subseteq A^{*-1}(0).$$

Apparently, if A satisfies property (N) at 0, then $A^{-1}(0)$ reduces A and

$$\sigma(A|_{A^{-1}(0)}) = \{0\}.$$

2.2 Results

As preliminaries, we prove:

Lemma 2.2.1 *If $Q = Q^2 \in B(\mathcal{H})$, then for Q to be self-adjoint it is sufficient that Q or $I - Q$ satisfies property (N) at 0.*

Proof. If $Q = Q^2$ satisfies property (N) at 0, then

$$Q(I - Q)x = 0 \Longrightarrow Q^*(I - Q)x = 0$$

for all $x \in \mathcal{H}$. Hence

$$Q^* = Q^*Q \Longrightarrow Q = Q^*.$$

Similarly, if $I - Q$ satisfies property (N) at 0, then

$$(I - Q)Qx = 0 \Longrightarrow (I - Q)^*Qx = 0 \text{ for all } x \in \mathcal{H} \Longrightarrow Q = QQ^*.$$

Once again, $Q = Q^*$.$\square$

Evidently, the condition of the lemma is necessary too.

Lemma 2.2.2 *$T \in B(\mathcal{H})$ is polar (of some order $k \in \mathbf{N}$) if and only if T^* is polar (of order k).*

Proof. This is an immediate consequence of the fact that if $\lambda - T$ is Drazin invertible with Drazin inverse S, then

$$(\lambda - T)^k S(\lambda - T) - (\lambda - T)^k = 0 = (\lambda - T)S - S(\lambda - T), \;\; S(\lambda - T)S = S$$

$$\begin{aligned} \iff \quad & (\lambda - T)^* S^* = S^*(\lambda - T)^*, (\lambda - T)^{*k} S^*(\lambda - T)^* - (\lambda - T)^{*k} = 0, \\ & S^*(\lambda - T)^* S^* = S^*. \;\; \square \end{aligned}$$

The following theorem subsumes most of the extant results on the determination of points $\lambda \in \text{iso}\sigma(T)$ for which $P_T(\lambda)$ is self-adjoint: in the sequel,we shall apply the theorem to a wide variety of classes of Hilbert space operators to recover these results. We assume, without loss of generality, in the following that $\lambda = 0$, and write P_T for $P_T(0)$.

Theorem 2.2.1 *Let $T \in B(\mathcal{H})$. If $0 \in iso\sigma(T)$, then the following are equivalent:*
(i) T is polar of order k and T^k satisfies property (N).
(ii) $P_T^* = P_T$ and $P_T\mathcal{H} = T^{-k}(0)$.
(iii) $P_T^* = P_T$ and $P_T^{-1}(0) = T^k\mathcal{H}$.

Proof. $(i) \implies (ii)$. If T is polar of order k and S denotes its Drazin inverse, then $P_T = I - ST$, $I - P_T$ is idempotent and

$$P_T\mathcal{H} = (I - ST)\mathcal{H} = T^{-k}(0).$$

Since, Lemma 2.2, T polar implies T^* polar (of the same order k), the same (also) holds with T^* in place of T and S^* in place of S, so that

$$P_{T^*}\mathcal{H} = (I - S^*T^*)\mathcal{H} = (T^k)^{*-1}(0).$$

This, taken along with the hypothesis that T satisfies property (N), implies

$$\begin{aligned} & (I - P_{T^*})^{-1}(0) = P_{T^*}\mathcal{H} = (T^k)^{*-1}(0) = T^{*-k}(0) \\ \supseteq \;\; & T^{-k}(0) = (T^k)^{-1}(0) = P_T\mathcal{H} = (I - P_T)^{-1}(0), \end{aligned}$$

which is property (N) for $I - P_T$. Hence, Lemma 2.1, $P_T = P_T^*$, giving us (ii).
$(ii) \Longrightarrow (iii)$. If (ii) is satisfied, then

$$\begin{aligned} & \mathcal{H} = P_T\mathcal{H} \oplus (I - P_T)\mathcal{H} = T^{-k}(0) \oplus P_T^{-1}(0) \\ \Longrightarrow \quad & T^k\mathcal{H} = 0 \oplus T^k P_T^{-1}(0) = P_T^{-1}(0). \end{aligned}$$

Since already $P_T^* = P_T$, $(ii) \Longrightarrow (iii)$.
$(iii) \Longrightarrow (i)$. If (iii) is satisfied, then

$$\begin{aligned} & P_T^{-1}(0) = T^k\mathcal{H} \Longrightarrow \{0\} = P_T T^k \mathcal{H} \Longrightarrow P_T\mathcal{H} = T^{-k}(0) \\ \Longrightarrow \quad & \mathcal{H} = P_T\mathcal{H} \oplus P_T^{-1}(0) = T^{-k}(0) \oplus T^k\mathcal{H}, \end{aligned}$$

i.e., T is polar of order k at 0. Bringing in the fact that P_T is self-adjoint, we also have that

$$\begin{aligned} P_T^*\mathcal{H} &= (T^{*k})^{-1}(0) = T^{*-k}(0) \\ &= P_T\mathcal{H} = (T^k)^{-1}(0) = T^{-k}(0). \end{aligned}$$

Thus $(iii) \Longrightarrow (i)$.$\square$

2.3 Applications

We apply now Theorem 2.3 to classes of Hilbert space operators; the interested reader is referred to [7] and [4] for further information on these classes of operators.

(a). If $T \in B(\mathcal{H})$ is either hyponormal ($|T^*|^2 \leq |T|^2$), or p-hyponormal for some $0 < p \leq 1$ ($|T^*|^{2p} \leq |T|^{2p}$), or log-hyponormal ($\log |T^*|^2 \leq \log |T|^2$), or M-hyponormal ($||(T-\lambda)^*x||^2 \leq M||(T-\lambda)x||^2$ for every $x \in \mathcal{H}$, all complex λ and some $M \geq 1$), or totally paranormal ($||(T-\lambda)x||^2 \leq ||(T-\lambda)^2x||||x||$ for every $x \in \mathcal{H}$ and complex λ), or totally $*$-paranormal ($||(T-\lambda)^*x||^2 \leq ||(T-\lambda)^2x||||x||$ for every $x \in \mathcal{H}$ and complex λ), then T is simply polar and $(T-\lambda)^{-1}(0) = (T-\lambda)^{*-1}(0)$ for every $\lambda \in \text{iso}\sigma(T)$. Hence the Riesz projection $P_T(\lambda)$ is self-adjoint for every $\lambda \in \text{iso}\sigma(T)$ for all such T (see also [2], [4] and [14]).

(b). If $T \in B(\mathcal{H})$ is either w-hyponormal ($|\tilde{T^*}| \leq |T| \leq |\tilde{T}|$, where, for the polar decomposition $T = U|T|$ of T, $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$), or of class $\mathcal{A}$ ($|T|^2 \leq |T^2|$), or quasi-class $\mathcal{A}$ ($T^*(|T^2| - |T|^2)T \geq 0$), or paranormal ($||Tx||^2 \leq ||T^2x||||x||$ for all $x \in \mathcal{H}$), or $*$-paranormal ($||T^*x||^2 \leq ||T^2x||||x||$ for all $x \in \mathcal{H}$), then T is simply polar at every $\lambda \in \text{iso}\sigma(T)$ and $(T-\lambda)^{-1}(0) \subseteq (T-\lambda)^{*-1}(0)$ for every non-zero $\lambda \in \text{iso}\sigma(T)$. Hence the Riesz projection $P_T(\lambda)$ is self-adjoint for every $0 \neq \lambda \in \text{iso}\sigma(T)$ for all such T (see also [4], [8], [11] and [16]).

(c). If $T \in B(\mathcal{H})$ is (p,k)-quasihyponormal, $T^{*k}(|T|^{2p} - |T^*|^{2p})T^k \geq 0$, for $0 < p \leq 1$ and some positive integer k, then $T = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} \overline{T^k\mathcal{H}} \\ T^{*-k}(0) \end{pmatrix}$, where $T_1 \in \mathcal{A}$ and T_2 is k-nilpotent [12]. (p,k)-quasihyponormal operators T are simply polar at points $0 \neq \lambda \in \text{iso}\sigma(T)$ and k-polar at $0 \in \text{iso}\sigma(T)$. Furthermore, the inclusion $(T-\lambda)^{-1}(0) \subseteq (T-\lambda)^{*-1}(0)$ holds for every non-zero $\lambda \in \text{iso}\sigma(T)$. Hence the Riesz projection $P_T(\lambda)$ is self-adjoint for every $0 \neq \lambda \in \text{iso}\sigma(T)$ and all (p,k)-quasihyponormal T. The inclusion $(T-\lambda)^{-1}(0) \subseteq (T-\lambda)^{*-1}(0)$ fails for $\lambda = 0$. To see this, let T have the upper triangular representation above, where $T_1 \in \mathcal{A}$ is invertible; then $T^{-k}(0) = \{x_1 \oplus x_2 : x_1 = T_1^{-1}Xx_2\}$ and $T^{*-k}(0) = \{0 \oplus x_2\}$. Evidently, $P_T(0) \neq P_T^*(0)$ (see also [4], [12] and [15]).

(d). An operator $T \in B(\mathcal{H})$ is normaloid if its norm equals its spectral radius; T is totally hereditarily normaloid, $T \in \mathcal{THN}$, if every part of T (i.e., its restriction to a closed invariant subspace), as well as the inverse of every invertible part (whenever it exists), is normaloid. The class of $\mathcal{THN}$ operators properly contains the class of paranormal (hence also, hyponormal, p-hyponormal, w-hyponormal and class $\mathcal{A}$) operators, but is properly contained in the class of normaloid operators (see [3]). $\mathcal{THN}$ operators T are simply polar at every $\lambda \in \text{iso}\sigma(T)$ [3], but may fail to satisfy $(T-\lambda)^{-1}(0) \subseteq (T-\lambda)^{*-1}(0)$. However, if $(T-\lambda)^{-1}(0) \subseteq (T-\lambda)^{*-1}(0)$ at a point $\lambda \in \text{iso}\sigma(T)$ for a $\mathcal{THN}$ operator T, then the Riesz projection $P_T(\lambda)$ is self-adjoint [4] (Corollary 2.1).

(e). An operator $T \in B(\mathcal{H})$ is a $wF(p,r,q)$ operator for some $p > 0$, $r \geq 0$ and $q \geq 1$ if $(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}$ and $|T|^{2(p+r)(1-\frac{1}{q})} \geq$

$(|T|^p|T^*|^{2r}|T|^p)^{1-\frac{1}{q}}$. For an operator $T \in wF(p,r,q)$ with polar decomposition $T = U|T$, the operator $T_{p,r} = |T|^pU|T|^r$ is m-hyponormal for $0 < m = \min\{\frac{1}{\mathrm{q}}, \max\{\frac{\mathrm{p}}{\mathrm{p+r}}, 1-\frac{1}{\mathrm{q}}\}\}$ [17]. Furthermore, T is subscalar if and only if $T_{p,r}$ is subscalar [18] (Lemma 3.3). Since m-hyponormal operators are subscalar [6], operator $T \in wF(p,r,q)$ are subscalar, and hence satisfy the property that $H_0(T-\lambda) = (T-\lambda)^{-t}(0)$ for some integer $t \geq 0$ and all complex λ [1] (p 175). Consequently, such operators T are polar at points $\lambda \in \mathrm{iso}\sigma(T)$ [5] (Example 2.1). Finally, since the non-zero eigenvalues of a $wF(P,r,q)$ operator are normal eigenvalues of the operator [17], the Riesz projection corresponding to a non-zero isolated point of the spectrum of a $wF(p,r,q)$ operator is self-adjoint [19] (Theorem 2.1).

It is our pleasure to thank Robin Harte for his help with the preparation of this note.

2.4 References

[1] P. Aiena, *Fredholm and Local Spectral Theory with Applications to Multipliers*, Kluwer, 2004.

[2] M. Chō and Y.M. Han, *Riesz idempotents and algebraically M-hyponormal operators*, Integr. Equat. Oper. Th. **53**(2005), 311-320.

[3] B. P. Duggal, S. V. Djordjević and C. S. Kubrusly, *Hereditarily normaloid contractions*, Acta Sci. Math. (Szeged) **71**(2005), 337-352.

[4] B. P. Duggal, *Riesz projections for a class of Hilbert space operators*, Lin. Alg. Appl. **407**(2005), 140-148.

[5] B. P. Duggal, *Hereditarily polaroid operators, SVEP and Weyl's theorem*, J. Math. Anal. Appl. **340**(2008), 366–373.

[6] B. P. Duggal, Finite intertwinings and subscalarity, Operators and Matrices **4**(2010), 257-271.

[7] T. Furuta, Invitation to Linear Operators, Taylor and Francis, London, 2001.

[8] Y.M. Han, J.I. Lee and D. Wang, *Riesz idempotents and Weyl's theorem for w-hyponormal operators*, Integr. Equat. Oper. Th. **53**(2005), 51-60.

[9] Robin E. Harte, Invertibility and Singularity, Dekker, 1988.

[10] H. G. Heuser, Functional Analysis, John Wiley and Sons (1982).

[11] In Ho Jeon and In Hyun Kim, *On operators satisfying* $T^*|T^2|T \geq T^*|T|^2T$, Lin. Alg. Appl. **418**(2006), 854-862.

[12] I. H. Kim, *On (p,k)-quasihyponormal operators*, Math. Inequal. Appl. **7**(2004), 629-638.

[13] M. Mbekhta, *Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux*, Glasgow Math. J. **29** (1987), 159-175.

[14] J. G. Stampfli, *Hyponormal operators*, Pacific J. Math. **12**(1962), 1453-1458.

[15] K. Tanahashi, A. Uchiyama and M. Chō, *Isolated points of the spectrum of (p,q)-quasihyponormal operators*, Linear Alg. Appl. **382**(2004), 221-229.

[16] A. Uchiyama, *On the isolated points of the spectrum of paranormal operators*, Integr. Equat. Oper. Th. **55**(2006), 145-151.

[17] C. Yang and J. Yuan, *Spectrum of class $wF(p,r,q)$ operators*, Acta Sci. Math. (Szeged) **71**(2005), 767-779.

[18] C. Yang and Y. Zhao, *On class $wF(p,r,q)$ operators and quasisimilarity*, J. Inequalities in Pure and Applied Math. **8**(2007), Article 90, 7pp.

[19] Jiangtao Yuan and Zongsheng Gao, *Spectrum of class $wF(p,r,q)$ operators*, J. Inequalities and Appl. Volume 2010, Article ID 27195, 10 pages.

3

Upper estimates for one-sided trigonometric approximation

Jorge Bustamante[1]

Abstract We present a constructive proof for one-sided approximation by trigonometric polynomials of a bounded function in the L_p norm. The estimates are given in term of the average moduli of smoothness. The only new results here are the estimates of the constants related with the errors.:

Keywords: One-sided approximation, rate of convergence, trigonometric approximation, moduli of smoothness.

[1]Faculta de Ciencias Físico Matemáticas, Benemérita Universidad Autónoma de Puebla, jbusta@fcfm.buap.mx

3.1 Introduction

Throughout the paper $L_p[0, 2\pi]$ $(1 \leq p < \infty)$ is the family of all real 2π-periodic function f with the usual norm

$$\|f\|_p = \left(\int_0^{2\pi} \mid f(t) \mid^p dt \right)^{1/p}.$$

and $\mathbb{T}_n$ is the family of all trigonometric polynomials of degree not greater than n.

For a function $f \in L_p[0, 2\pi]$ bounded from below (above), the best lower (upper) trigonometric approximation of order n is defined by

$$E_n^-(f)_p \;=\; \inf \{ \|T - f\|_p \, : T \in \mathbb{T}_n, \;\; f \geq T \; a.e\}$$

$$(E_n^+(f)_p \;=\; \inf \{ \|T - f\|_p \, : T \in \mathbb{T}_n, \;\; T \geq f \; a.e.\}).$$

These one-sided approximations in L_p spaces have been studied by different authors.

As usual, the difference of order r with step h is defined by

$$\Delta_h^r f(x) \;=\; \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\,(x + kh)\,.$$

For a bounded 2π-periodic function f we consider the local modulus

$$\omega_r(f, x, t) = \; \sup \{ \mid \Delta_h^r f(y) \mid : \; y, y + rh \in [x - rt, x + rt], \; \mid h \mid \leq t \}. \quad (3.1)$$

For $1 \leq p < \infty$ we set

$$\tau_k(f, t)_p = \|\omega_k(f, \cdot, t)\|_p = \left(\int_0^{2\pi} |\omega_k(f, x, t)|^p \; dx \right)^{1/p}. \quad (3.2)$$

Moduli of smoothness similar to (3.2) have been consider by several authors (for instace, see [1], [2], [3], [4] and[5]). In particular Popov and Andreev proved that for each r, there exists a constant $C(r)$ such that, for any bounded 2π-periodic function f

$$E_n^{\pm}(f)_p \;\leq\; C(r) \;\; \tau_r \left(f, \frac{2\pi}{n} \right)_p. \quad (3.3)$$

Some of the known proofs for these kind of inequalities use splines as an intermediate approximation (see [1]). In [5] Sadrin used the Jackson kernels to construct polynomials for one sided approximation. He found an upper estimate for one-sided approximation with a modulus similar to (3.2), but without the condition $\mid h \mid\leq t$ in (3.1). In this paper we will refine some of the arguments used by Sadrin to obtain an estimate of the constant $C(r)$ in (3.3). In particular, we show that in (3.3) we can take

$$C(r) \;=\; \frac{75}{32}\,\pi^{r+4}.$$

In Section 3.2 we present some auxiliary result. The main theorem is proved in Section 3.3.

3.2 Some properties of the moduli

Proposition 3.2.1 *Let f be a 2π-periodic bounded function and fix $r \in \mathbb{N}$. Then for any $u, t \in [0,\pi]$ and $k, m \in \mathbb{N}$*

(i) *For any $t > 0$, $\omega_k(f,\cdot,t)$ is a 2π-periodic function.*

(ii) *If $0 < u < t$ and $x \in [-\pi,\pi]$, then*

$$\omega_r(f,x,u) \leq \omega_r(f,x,t).$$

(iii) *If $u, t, x \in [-\pi,\pi]$, then*

$$\mid \Delta_t^r f(x) \mid\leq \omega_r(f, x+t, \mid u \mid) + \omega_r(f, x+ \operatorname{sign}(ut)u, \mid t \mid) \tag{3.4}$$

and

$$\omega_r(f,x,\mid t \mid) + \omega_r(f, x+\operatorname{sign}(ut)\,u, \mid t \mid) + \omega_r(f,x+t,\mid u \mid)$$

$$\leq 3\omega_r(f,x,2\mid t \mid) + 2\omega_r(f,x,2\mid u \mid). \tag{3.5}$$

Proof. (i) and (ii) follows directly from the definition.

(iii) Case 1. Assume first that $\mid t \mid\leq\mid u \mid$. If $t \geq 0$, then

$$x+t-r\mid u \mid\leq x-(r-1)\mid u \mid\leq x \leq x+rt \leq x+t+r\mid u \mid .$$

Thus
$$| \Delta_t^r f(x) | \leq \omega_r(f, x+t, | u |).$$

If $t < 0$, then

$$x + t - r \mid u \mid \leq x + rt \leq x \leq x + t + r \mid u \mid .$$

Thus
$$| \Delta_t^r f(x) | \leq \omega_r(f, x+t, | u |).$$

Case 2. Now, assume that $| u | < | t |$.
If $t \geq 0$ and $0 \leq u$, then

$$x + u - r \mid t \mid \leq x - (r-1) \mid t \mid \leq x \leq x + rt \leq x + u + r \mid t \mid .$$

Thus
$$| \Delta_t^r f(x) | \leq \omega_r(f, x+u, | t |).$$

If $t \geq 0$ and $u < 0$, then

$$x - u - r \mid t \mid \leq x - (r-1) \mid t \mid \leq x \leq x + rt \leq x - u + r \mid t \mid .$$

Thus
$$| \Delta_t^r f(x) | \leq \omega_r(f, x-u, | t |).$$

If $t < 0$ and $u \geq 0$, then

$$x - u - r \mid t \mid \leq x + rt \leq x \leq x - u + r \mid t \mid .$$

Thus
$$| \Delta_t^r f(x) | \leq \omega_r(f, x-u, | t |).$$

If $t < 0$ and $u < 0$, then

$$x + u - r \mid t \mid \leq x + rt \leq x \leq x + u + r \mid t \mid .$$

Thus
$$| \Delta_t^r f(x) | \leq \omega_r(f, x+u, | t |).$$

(iv) Case 1. Assume $\mid t \mid \leq \mid u \mid$. If

$$x + t - r \mid u \mid \leq y, y + rh \leq x + t + r \mid u \mid,$$

then

$$x - 2r \mid u \mid \leq y, y + rh \leq x + 2r \mid u \mid .$$

Therefore

$$\omega_r(f, x + t, \mid u \mid) \leq \omega_r(f, x, 2 \mid u \mid)$$

If

$$x + \mathrm{sign}(ut)\, u - r \mid t \mid \leq y, y + rh \leq x + \mathrm{sign}(ut)\, u + r \mid t \mid,$$

then

$$x - 2r \mid u \mid \leq y, y + rh \leq x + 2r \mid u \mid .$$

Therefore

$$\omega_r(f, x + \mathrm{sign}(ut)\, u, \mid t \mid) \leq \omega_r(f, x, 2 \mid u \mid)$$

Case 2. Now, assume that $\mid u \mid < \mid t \mid$.
If

$$x + t - r \mid u \mid \leq y, y + rh \leq x + t + r \mid u \mid,$$

then

$$x - 2r \mid t \mid \leq y, y + rh \leq x + 2r \mid t \mid .$$

Therefore

$$\omega_r(f, x + t, \mid u \mid) \leq \omega_r(f, x, 2 \mid t \mid)$$

If

$$x + \mathrm{sign}(ut)\, u - r \mid t \mid \leq y, y + rh \leq x + \mathrm{sign}(ut)\, u + r \mid t \mid,$$

then

$$x - 2r \mid t \mid \leq y, y + rh \leq x + 2r \mid t \mid .$$

Therefore

$$\omega_r(f, x + \mathrm{sign}(ut)\, u, \mid t \mid) \leq \omega_r(f, x, 2 \mid t \mid)$$

Proposition 3.2.2 *Let f be a 2π-periodic bounded function and fix $r \in \mathbb{N}$. If $x \in [-\pi, \pi]$, $t > 0$ and $n \in \mathbb{N}$, then*

$$\omega_r(f, x, nt) \leq n^r \max_{-n\leq k\leq n} \omega_r(f, x + krt, t)$$

and, for $p \in [1, \infty)$,

$$\tau_r(f, nt)_p \leq\ (2n + 1)n^r\, \tau_r(f, t)_p.$$

Proof. Fix $\varepsilon > 0$, y and h such that $| \ h \ |\leq nt$,

$$x - rnt \ \leq \ y, y + rh \ \leq \ x + rnt,$$

and $\omega_r(f, nt) < \varepsilon+ \mid \Delta^r_h f(y) \mid$.

We will consider the case $h > 0$. Set $h = ns$. It can be proved (by induction with respect to r) that

$$\Delta^r_{ns} f(y) \ = \sum_{i_1=0}^{n-1} \cdots \sum_{i_r=0}^{n-1} \Delta^r_s f(y + s(i_1 + i_2 + \cdots + i_r))$$

In order to estimate $\Delta^r_s f(y + s(i_1 + i_2 + \cdots + i_r))$, if $s > 0$, fix an integer k such that

$$x + krt \leq y + s \sum_{j=1}^{r} i_j < x + r(k + 1)t.$$

Since

$$x - rnt \leq y \leq y + s \sum_{j=1}^{r} i_j \leq y + sr(n - 1) = y + rh - sr < x + rnt,$$

and one has $-n \leq k < n$.

With k selected as above we obtain

$$x + r(k + 1)t - rt = x + krt \ \leq y + s \sum_{j=1}^{r} i_j < y + s \sum_{j=1}^{r} i_j + rs$$

$$\leq x + (k+1)rt + rt.$$

Therefore

$$\mid \Delta_s^r f(y + s(i_1 + i_2 + \cdots + i_r)) \mid \leq \omega_r(f, x + r(k+1)t, t).$$

If $s < 0$, then

$$x - rnt < y + rn - sr \leq y + sr(n-1) \leq y + s\sum_{j=1}^{r} i_j \leq y \leq x + rnt.$$

Thus we can fix an integer k $(-n \leq k < n)$, such that

$$x + krt < y + s\sum_{j=1}^{r} i_j \leq x + r(k+1)t.$$

In this case

$$x + krt - rt \ \leq y + s\sum_{j=1}^{r} i_j + rs < y + s\sum_{j=1}^{r} i_j \leq x + (k+1)rt.$$

Therefore

$$\mid \Delta_s^r f(y + s(i_1 + i_2 + \cdots + i_r)) \mid \leq \omega_r(f, x + rkt, t).$$

From the arguments give above one has

$$\omega_r(f, x, nt) \leq \varepsilon + n^r \max_{-n \leq k \leq n} \omega_r(f, x + krt, t),$$

and it is sufficient to prove the first inequality.

For the second we have

$$\tau_r(f, nt)_p = \left(\int_{-\pi}^{\pi} (\omega_r(f, x, nt))^p \ dx \right)^{1/p}$$

$$\leq \ n^r \left(\int_{-\pi}^{\pi} \left(\sum_{k=-n}^{n} \omega_r(f, x + krt, t) \right)^p \ dx \right)^{1/p}$$

$$\leq n^r \sum_{k=-n}^{n} \left(\int_{-\pi}^{\pi} (\omega_r(f, x + krt, t))^p \, dx \right)^{1/p}$$

$$= (2n+1)n^r \left(\int_{-\pi}^{\pi} (\omega_r(f, x, t))^p \, dx \right)^{1/p} . \square$$

3.3 Main result

For $s, m \in \mathbb{N}$ the Jackson kernel is defined by

$$I_{s,m}(t) = \gamma_{s,m} \left(\frac{\sin(mt/2)}{\sin(t/2)} \right)^{2s}$$

where $\gamma_{s,m}$ is taken from the condition

$$\int_{-\pi}^{\pi} I_{s,m}(t) dt = 1.$$

It is known that $I_{s,m}$ is a trigonometric polynomial of degree $s(m-1)$.

Theorem 3.3.1 *If $1 \leq p < \infty$ and $f \in M$, then for any $r, n \in \mathbb{N}$ one has*

$$E_n^-(f)_p \leq \frac{75}{32} \pi^{r+4} \tau_r \left(f, \frac{2\pi}{n} \right)_p .$$

Proof. Let s the least integer satisfying $r + 3 \leq 2s$ and set $m = 1 + [n/s]$ ($[z]$ is the integer part of z).

Define

$$T_n(f, x) = - \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} U(f, x, t, u) I_{r,m}(t) I_{r,m}(u) dt du.$$

where

$$U(f, x, t, u) = \omega_k(f, x + \text{sign}(ut)\, u, |\, t \,|) + \omega_k(f, x + t, |\, u \,|) + \Delta_t^r f(x) - f(x).$$

Notice that $T_n \in \mathbb{T}_{s(m-1)}$ and $s(m-1) \le n$.

Taking into account (3.4) and (3.5) we know that

$$
\begin{aligned}
0 &\le f(x) - T_n(x) \\
&= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} [\Delta_t^r f(x) + \omega_r(f, x + \operatorname{sign}(ut)\, u, \mid t \mid) \\
&\qquad\qquad + \omega_r(f, x+t, \mid u \mid)\} I_{r,m}(t) I_{r,m}(u) dt du \\
&\le \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} [\omega_r(f, x, \mid t \mid) + \omega_r(f, x + \operatorname{sign}(ut)\, u, \mid t \mid) \\
&\qquad\qquad + \omega_r(f, x+t, \mid u \mid)\} I_{r,m}(t) I_{r,m}(u) dt du \\
&\le \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} [3\omega_r(f, x, 2 \mid t \mid) + 2\omega_r(f, x, 2 \mid u \mid)\} I_{r,m}(t) I_{r,m}(u) dt du \\
&= 10 \int_0^{\pi} \omega_r(f, x, 2t) I_{r,m}(t) dt.
\end{aligned}
$$

Taking the L_p-norm, changing the order of integration and using Prop. 3.2.2, we obtain

$$
\begin{aligned}
E_n^-(f)_p &\le 10 \int_0^{\pi} \tau_r(f, 2t)_p I_{r,m}(t) dt \\
&\le 10 \tau_r\left(f, \frac{2\pi}{n}\right)_p \int_0^{\pi} \left(3 + \frac{2nt}{\pi}\right)\left(1 + \frac{nt}{\pi}\right)^r I_{r,m}(t) dt.
\end{aligned}
$$

Let us estimate the last term. Taking into account that $2v/\pi \le \sin v$, for $v \in (0, \pi/2]$, one has

$$
\begin{aligned}
\int_{-\pi}^{\pi} \left(\frac{\sin(mt/2)}{\sin(t/2)}\right)^{2s} dt &= 2\int_0^{\pi} \left(\frac{\sin(mt/2)}{\sin(t/2)}\right)^{2s} dt \\
\ge 2 \int_0^{\pi/m} \left(\frac{(2/\pi)(mt/2)}{t/2}\right)^{2s} dt &= 2\left(\frac{2m}{\pi}\right)^{2s} \frac{\pi}{m}.
\end{aligned}
$$

Hence

$$
\gamma_{s,m} \le \left(\frac{\pi}{2m}\right)^{2s} \frac{m}{2\pi}. \tag{3.6}
$$

Since $r \le 2s-3$, for $1 \le i \le r+1$,

$$\begin{aligned}
&\int_0^{\pi} t^i \left(\frac{\sin(mt/2)}{\sin(t/2)}\right)^{2s} dt \\
&= \int_0^{\pi/m} t^i \left(\frac{\sin(mt/2)}{\sin(t/2)}\right)^{2s} dt + \int_{\pi/m}^{\pi} t^i \left(\frac{\sin(mt/2)}{\sin(t/2)}\right)^{2s} dt \\
&\le \int_0^{\pi/m} t^i \left(\frac{m\sin(t/2)}{\sin(t/2)}\right)^{2s} dt + \int_{\pi/m}^{\pi} t^i \left(\frac{\pi}{t}\right)^{2s} dt \\
&\le \frac{m^{2s}}{i+1}\left(\frac{\pi}{m}\right)^{i+1} + \frac{\pi^{2s}}{2s-i-1}\left(\frac{m}{\pi}\right)^{2s-i-1} \le \frac{3}{2} m^{2s-i-1}\pi^{i+1}.
\end{aligned}$$

From the last inequality and (3.6) we obtain

$$\int_0^{\pi} t^i I_{s,m}(t)dt \;\le\; \left(\frac{\pi}{2m}\right)^{2s} \frac{m}{2\pi}\frac{3}{2} m^{2s-i-1}\pi^{i+1} = \left(\frac{\pi}{2}\right)^{2s}\frac{3}{4}\frac{\pi^i}{m^i}.$$

Then,

$$\begin{aligned}
&\int_0^{\pi} \left(3+\frac{2mt}{\pi}\right)\left(1+\frac{mt}{\pi}\right)^r I_{s,m}(t)dt \\
&= \sum_{i=0}^{r}\binom{r}{i}\int_0^{\pi}\left(3+\frac{2mt}{\pi}\right)\left(\frac{mt}{\pi}\right)^i I_{s,m}(t)dt \\
&= 3\sum_{i=0}^{r}\binom{r}{i}\left(\frac{m}{\pi}\right)^i \int_0^{\pi} t^i I_{s,m}(t)dt + 2\sum_{i=0}^{r}\binom{r}{i}\left(\frac{m}{\pi}\right)^{i+1}\int_0^{\pi} t^{i+1} I_{s,m}(t)dt \\
&\le \frac{3}{2} + \frac{3}{4}\left(\frac{\pi}{2}\right)^{2s}\left(3\sum_{i=1}^{r}\binom{r}{i}\left(\frac{m}{\pi}\right)^i\frac{\pi^i}{m^i} + 2\sum_{i=0}^{r}\binom{r}{i}\left(\frac{m}{\pi}\right)^{i+1}\frac{\pi^{i+1}}{m^{i+1}}\right) \\
&= \frac{3}{2} + \frac{3}{4}\left(\frac{\pi}{2}\right)^{2s}(52^r-3) \le \frac{3}{2} + \frac{3}{4}\left(\frac{\pi}{2}\right)^{r+4}(52^r-3) \le \frac{15}{64}\pi^{r+4},
\end{aligned}$$

since $2s \le r+4$.

From the arguments given above we obtain

$$E_n^-(f)_p \;\le\; \frac{75}{32}\,\pi^{r+4}\tau_r\left(f,\frac{2\pi}{n}\right)_p . \square$$

3.4 References

[1] A. Andreev, V. A. Popov and B. Sendov, Jackson-type theorems for best one-side approximations by trigonometrical polynomials and splines, Mathematical Notes, 1979, 889-896.

[2] V. A. Popov, Local approximation of a function, Math. Notes, 1975, 213-221.

[3] V. A. Popov, Converse theorems for one-sided trigonometrical approximation, Compt. Rend. Acad. Bulg. Sci., 30 (1977), 1529-1532.

[4] V. A. Popov and A. S. Andreev, Stechkin's type theorems for one-sided trigonometrical and spline approximation, Compt. Rend. Acad. Bulg. Sci., 31 (1978), 151-154.

[5] A. Yu. Shadrin, Orders of one-sided approximations of functions in L_p-metric, Analysis Math., 12 (1986), 175-184.

4

Where algebra and topology meet: a cautionary tale

Robin Harte[1] **and Dragana Cvetković- Ilić**[2]

Abstract: In a sense the Kuratowski conditions reduce topology to algebra. In another sense a simple property of Banach algebras ushers in a curious topology for rings.

4.1 Algebra

A Banach algebra is many things: a failed C* algebra, an over specialised locally convex algebra: but first and foremost a *ring* - it has addition and multiplication, which fit together like ordinary arithmetic. If you will forgive us, our rings always have *identity* 1 - multiplicatively neutral:

1.1 $$1x = x = x1 \; .$$

Similarly the *zero* 0 is additively neutral:

1.2 $$0 + x = x = x + 0 \; .$$

[1]Trinity College Dublin, email: rharte@maths.tcd.ie
[2]University of Niš, email: gagamaka@ptt.yu

So you can imagine 1 and 0 meeting at conferences, complaining how nobody seems to appreciate them, think they do nothing. Which is of course to underestimate zero: when she is turned loose on the multiplication

1.3 $$0x = 0 = x0\ :$$

total loss of information, there is no way back when you have been multiplied by zero. Now everything in a ring A must choose between 0 and 1 ; the friends of 1 , the best friends of 1 , are known as the *invertibles*:

1.4 $$yx = 1 = xy\ .$$

Since all those inverses are unique - no single x can have two different inverses y - we can give them names, and write - pace **Oliver Heaviside** -

1.5 $$y = x^{-1}\ .$$

We would also like to write

1.6 $$A^{-1} \subseteq A$$

for the set of those invertibles, known as the *invertible group* - you can easily write down the inverse of the product of two invertibles,

1.7 $$(vu)^{-1} = u^{-1}v^{-1}\ ,$$

the inverse of the inverse of an invertible, and the inverse of the identity

So that is the ring theory: a *linear algebra* also admits multiplication by ordinary numbers. There are real and there are complex algebras: thanks to the identity the scalars are in effect a subring.

4.2 Topology

So that is the algebra: the *topology* is smuggled in through the concept of *length*. Everything $x \in A$ has its "norm" $\|x\| \in \mathbf{R}$: this is a positive real number, possibly zero (but only for $x = 0$), and subject to the *triangle inequality*:

2.1 $$\|x + y\| \leq \|x\| + \|y\| \ ;$$

analagously

2.2 $$\|xy\| \leq \|x\| \ \|y\| \ ,$$

and $\|1\| = 1$. Length brings distance: the distance from x to y is the length of $y - x$,

2.3 $$\mathrm{dist}(x, y) = \|y - x\| \ .$$

Distance in turn brings *topology*: limits, convergence and continuity. We shall document the *neighbourhoods* of a point, and the *closure* of a subset. Specifically $U \subseteq A$ is to be a "neighbourhood" of $x \in A$, written $U \in \mathrm{Nbd}(x)$, if it completely insulates it from the outside world: there has to be a (strictly) positive real number $\varepsilon > 0$ for which

2.4 $$\mathrm{dist}(x, x') < \varepsilon \Longrightarrow x' \in U \ .$$

Now $x \in A$ is to be in the "closure" of $K \subseteq A$ precisely when the complement of K is not a neighbourhood of x: $x \in \mathrm{cl}(K)$ means that, no matter how strict your tolerance $\varepsilon > 0$ there will be $x' \in A$ for which

2.5 $$x \in K \text{ and } \mathrm{dist}(x, x') < \varepsilon \ ,$$

Closure obeys the *Kuratowski axioms*:

$[Ku_0]$ $$\mathrm{cl}(\emptyset) = \emptyset \ ;$$

$[Ku_1]$ $$K \subseteq \mathrm{cl}(K) \ ;$$

$[Ku_2]$ $$K \subseteq H \Longrightarrow \mathrm{cl}(K) \subseteq \mathrm{cl}(H) \ ;$$

$[Ku_3]$ $$\mathrm{cl}(\mathrm{cl}(K)) \subseteq \mathrm{cl}(K) \ ;$$

$[Ku_4]$ $$\mathrm{cl}(K_\cup H) \subseteq \mathrm{cl}(K)_\cup \mathrm{cl}(H) \ .$$

4.3 Neumann series

We are nearly at the end of the flatpack labelled "Banach algebra"; there is a sort of guarantee at the bottom of the box which reads

3.1 *anything that deserves to converge does converge .*

Formally if for every neighbourhood $U \in \mathrm{Nbd}(0)$ there is $N \in \mathbf{N}$ for which

3.2 $$n \geq m \geq N \Longrightarrow x_n - x_m \in U$$

then there is $y \in A$ for which

3.3 $$n \geq N \Longrightarrow x_n - y \in U \ .$$

We put this to work rather quickly. Suppose $\|x\| < 1$: then on the one hand [7], with no need for the guarantee, x^n *deserves to converge, and does converge, to 0* :

3.4 $$x^n \to 0 \text{ as } n \to \infty \ .$$

For the same reason it is true [7], more delicately,

3.5 $$1 + x + x^2 + \ldots + x^n \text{ deserves to converge .}$$

Since evidently

3.6 $$(1 - x)(1 + x + \ldots + x^n) = 1 - x^{n+1} \to 1 - x0 = 1 - 0 = 1 \ ,$$

it is clear that

3.7 $$1 + x + \ldots + x^n \to y \Longrightarrow (1 - x)y = 1 = y(1 - x) \ :$$

With the help of our guarantee (3.1) it follows

3.8 $$\|x\| < 1 \Longrightarrow 1 - x \in A^{-1} \ .$$

This simple observation is at the root of some of the simpler properties of a Banach algebra. We extend its reach to a simple consequence of membership of the closure of an arbitrary set $K \subseteq A$: if $x \in \mathrm{cl}(K)$ then

3.9 $$\forall y \in A \ , \ \exists x' \in K \ : 1 - y(x - x') \in A^{-1}.$$

3.10 **Problem** *In a Banach algebra, is the condition* (3.9) *equivalent to* $x \in \mathrm{cl}(K)$*?*

4.4 Algebraic closure

When you cannot establish an equality, you can always make a *Definition*: in an arbitrary ring A declare $x \in Cl(K)$ to be in the *algebraic closure* of $K \subset A$ iff (3.9) holds. If you feel that (3.9) looks a little "one sided", do not forget [7] the *Jordan lemma*:

4.1 $$1 - y(x - x') \in A^{-1} \Longleftrightarrow 1 - (x - x')y \in A^{-1} .$$

In return for such presumption, we are morally obliged to test (3.9) for *Kuratowski*.

[Ku$_0$], [Ku$_1$] and [Ku$_2$] are rather easy [3] and need not detain us; the argument for [Ku$_3$] says a lot about a certain kind of mathematical argument, and also about the algebraic closure.

Towards [Ku$_3$] therefore suppose $x \in A$ is in the closure of the closure of $K \subseteq A$: then if $y \in A$ is arbitrary there is x' in the closure of K for which

$$1 - y(x - x') \in A^{-1} \text{ invertible },$$

and if $z \in A$ is arbitrary there is x'' in K for which

$$1 - z(x' - x'') \in A^{-1} \text{ invertible }.$$

Now - **Edward de Bono** Lateral Thinking moment - write something down:

$$1 - y(x - x'') = (1 - y(x - x')) - y(x' - x'') .$$

Here this first term $1 - y(x - x') = u^{-1}$ is invertible, therefore can be divided through:

$$1 - y(x - x'') = (1 - y(x - x'))(1 - uy(x' - x'')) .$$

At this point we look very hard at the provenance of $x'' \in K$, which came from $z \in A$. Choosing $z = uy$ (method of undetermined coefficients!) gives

$$1 - uy(x' - x'') = 1 - z(x' - x'') = v^{-1} \text{ invertible } :$$

thus

$$1 - y(x - x'') = u^{-1}v^{-1} = (vu)^{-1} \in A^{-1} \bullet$$

4.5 Algebraic foreclosure

Unfortunately now we hit the buffers: we are unable to verify [Ku$_4$]. Indeed there is [3] a counterexample:

5.1 $$A = \mathbf{C}^2 \ ; \ K = \{1\} \times \mathbf{C} \ ; \ H = \{-1\} \times \mathbf{C} \ ; \ a = (2,3)$$

gives

5.2 $$a \in Cl(K_\cup H) \ ; \ a \notin Cl(K) \ ; \ a \notin Cl(H) \ .$$

From this example it is clear that the solution of Problem 3.10 is negative. In spite of this the "algebraic closure" is out there: for example

5.3 $$A^{-1}_{left} \cap Cl(A^{-1}) = A^{-1} = A^{-1}_{right} \cap Cl(A^{-1}) \ :$$

for if $a \in A^{-1}_{left} \cap Cl(A^{-1}_{right})$ then there are $a' \in A$ and $a'' \in A^{-1}_{right}$ for which

5.4 $$a'a = 1 \ ; \ 1 - a'(a - a'') \in A^{-1} \ ,$$

taking $y = a'$ in (3.9). This means firstly that $a'' \in A^{-1}_{right} \cap A^{-1}_{left} = A^{-1}$, and secondly that $a' \in A^{-1}_{left} \cap A^{-1}_{right} = A^{-1}$, giving finally $a \in A^{-1}$ •

It then follows that for example

5.5 $$Cl(A^{-1}_{left}) = A \Longrightarrow A^{-1}_{right} = A^{-1} = A^{-1}_{left} \ .$$

Similar argument shows [6],[7] that anything in the closure of the invertibles which has a generalized inverse also has an invertible generalized inverse, and if it is in an abstract sense "Fredholm" then it is also [6],[7], in an abstract sense, "of index zero".

4.6 Rescue package

While the conditions $[Ku_0] - [Ku_3]$ justify the epithet "closure", in the sense that for example the "convex hull" of a set of vectors is a kind of closure, a proper theory of limits really rests on $[Ku_4]$. With hindsight we gave up too soon when we settled for the definition (3.9); the proper version was a whisker away:

Definition. *In a ring A, the element $x \in A$ will be said to be in the spectral closure of the subset $K \subseteq A$, written $x \in Cl(K)$, iff*

6.1 $$\forall \text{ finite } J \subseteq A\ ,\ \exists x' \in K\ : 1 - J(x - x') \subseteq A^{-1}\ .$$

With this we get [Ku_0]-[Ku_3] and also [Ku_4]. Indeed suppose $x \in A$ is in the closure of $K_\cup H$, and not in the closure of H: then there is (x'_J) in $K_\cup H$ for which

6.2 $$1 - J(x - x'_J) \subseteq A^{-1}\ (J \in \mathrm{Finite}(A))\ .$$

There is also at least one finite subset $L \subseteq A$ for which there does not exist $w \in H$ for which

6.3 $$1 - L(x - w) \subseteq A^{-1}\ ,$$

This does not rule out the possibility that each individual $y \in L$ satisfies (3.9). But looking at $J \cup L$, necessarily

$$x'_{J\cup L} \in K\ \bullet$$

In effect (x'_J) is a "generalized sequence", and $x''_J = x'_{J\cup L}$ a "generalized subsequence". While going to the extra trouble to replace (3.9) by (6.1) does not materially alter the discussion of the norm closure in Banach algebras, it does allow us to pass freely between "closure", "interior" and "neighbourhood", and use the language of point set topology. Thus $U \subseteq A$ is a neighbourhood of $0 \in A$ provided there exists finite $J \subseteq A$ for which

6.4 $$1 + JU \subseteq A^{-1}\ ;$$

$x = (x_n) \in A^{\mathbf{N}}$ converges to $y \in A$ provided that for every finite $J \subseteq A$ there is $N \in \mathbf{N}$ for which

6.5 $$n \geq N \Longrightarrow 1 + Jx_n \subseteq A^{-1}\ .$$

4.7 Housekeeping

For example this gives back the usual discrete topology on the ring of integers $\mathbf{Z}$:

7.1 $$K \subseteq \mathbf{Z} \Longrightarrow Cl(K) = K \ .$$

If A is a division ring then the closed sets are, apart from the whole space, the finite sets: if $K \subseteq A = A^{-1} \cup \{0\}$ then

7.2 $$\#K < \infty \Longrightarrow Cl(K) = K \ ; \ \#K \geq \infty \Longrightarrow Cl(K) = A \ .$$

Thus we do not get back the usual topology for real or complex scalars; in this new environment the solution to Problem 3.10 remains negative. In a general Banach algebra the norm topology is stronger than the algebraic: if $K \subseteq A$ then

7.3 $$\mathrm{cl}(K) \subseteq Cl(K) \ .$$

Generally the closure of the single point is its coset modulo the *radical*: if $x \in A$ is arbitrary then

7.4 $$Cl(\{x\}) = x + \mathrm{Rad}(A) \ :$$

where

7.5 $$\mathrm{Rad}(A) = \{a \in A : 1 - Aa \subseteq A^{-1}\} \ .$$

In particular it follows that for an arbitrary ring A

7.6 $$A \text{ separated } \Longleftrightarrow A \text{ semi simple} \ .$$

If A and B are rings then the cartesian product $A \times B = A \oplus B$ is also a ring, with coordinatewise addition and multiplication, and it is easily checked that the algebraic closure on this product gives the cartesian product of the topologies on the factor. In general addition and multiplication are jointly continuous, from $A \times A$ to A, with respect to the spectral topology. The invertible group A^{-1} is an open set, and inversion $z^{-1} : a \mapsto a^{-1}$ is continuous there:

7.7 $$A^{-1} = Int(A^{-1}) \ .$$

Indeed for arbitrary $x \in A^{-1}$

7.8 $$1 - x^{-1}(x - x') \in A^{-1} \Longrightarrow x' \in A^{-1} \ .$$

Curiously enough if A is a "topological ring", with separately continuous multiplication, then the topological analogue of (7.7) is equivalent to the comparison of topologies (7.3).

From (5.3) and (7.7) it follows that the topological boundary, relative to the algebraic closure, of the invertible group is disjoint from the semi invertibles $A^{-1}_{left} \cup A^{-1}_{right}$.

4.8 Nearly invertibles

We shall describe elements of $Cl(K)$ as being *nearly in* $K \subseteq A$: in particular the *nearly invertibles* form the set

8.1 $$Cl(A^{-1}) \ .$$

Evidently the nearly invertibles are closed under multiplication:

8.2 $$Cl(A^{-1})Cl(A^{-1}) \subseteq Cl(A^{-1}) \ .$$

By (5.3),

8.3 *nearly invertible and semi invertible implies invertible* .

Thus in particular, a nearly invertible implies a *left-right consistent*:, in the sense [5] that for arbitrary $b \in A$

8.4 $$ab \in A^{-1} \Longleftrightarrow ba \in A^{-1} \ .$$

Nearly invertibles with "generalized inverses" always have invertible generalized inverses: with

8.5 $$A^{\cap} = \{a \in A : a \in aAa\}$$

and

8.6 $$A^{\cup} = \{a \in A : a \in aA^{-1}a\} \ ,$$

we have

8.7 $$A^{\cap} \cap Cl(A^{-1}) \subseteq A^{\cup} \ ,$$

with equality iff

8.8 $$A^{\bullet} \subseteq Cl(A^{-1}) \ ,$$

where

8.9 $$A^{\bullet} = \{a \in A : a^2 = a\} \ .$$

Indeed [5]

$$a = aa'a \ , \ b \in A^{-1} \ , \ 1 - a'(a-b) = c^{-1} \in A^{-1}$$

implies

$$a = (aa')(bc) \in A^{\bullet}A^{-1} \subseteq A^{\cup} \ ;$$

conversely

$$A^{\cup} \subseteq A^{\bullet}A^{-1} = A^{-1}A^{\bullet} \ .$$

4.9 Fredholm theory

A homomorphism $T : A \to B$ is by definition a mapping that respects addition and multiplication, and for us preserves identity. It is easily checked that if $T : A \to B$ is a homomorphism then

9.1 $$T(A) = B \Longrightarrow T \text{ continuous} \ .$$

Since it we always have

9.2 $$T(A^{-1}) \subseteq B^{-1} ,$$

homomorphisms $T : A \to B$ generate *Weyl* and *Fredholm* elements of the departure ring A:

9.3 $$A^{-1} \subseteq A^{-1} + T^{-1}(0) \subseteq T^{-1}B^{-1} \subseteq A .$$

For example if $A = B(X)$ is the bounded operators on a Banach space X and $T : A \to B$ the Calkin homomorphism from A to $B = A/J$, with either $J = K(X)$ the compact operators or $J = K_0(X)$ the finite rank operators, then

9.4 $$T^{-1}B^{-1} = \{a \in A : \max(\dim a^{-1}(0), \dim X/a(X)) < \infty\}$$

gives back the classical "Fredholm operators", with finite dimensional null spaces and closed ranges of finite codimension, while

9.5 $$A^{-1} + T^{-1}(0) = \{a \in T^{-1}B^{-1} : \dim a^{-1}(0) = \dim X/a(X)\}$$

gives back the Fredholm operators "of index zero". If instead $A = C(\mathbf{D})$ is the continuous functions on the closed unit disc $\mathbf{D} \subseteq \mathbf{C}$, $B = C(\partial\mathbf{D})$ the functions on the unit circle $\mathbf{S} = \partial\mathbf{D} \subseteq \mathbf{C}$ and $T : A \to B$ the restriction mapping, then

9.6 $$T^{-1}B^{-1} = \{a \in A : a^{-1}(0) \cap \partial\mathbf{D} = \emptyset\}$$

consists of those functions on the disc which never vanish on the circle while

9.7 $$A^{-1} + T^{-1}(0) = \{a \in T^{-1}B^{-1} : \mathrm{index}(Ta) = 0\}$$

consists of those whose restrictions have index or "winding number" zero. It is plausible that

nearly invertible Fredholm $\Longleftrightarrow$ *Weyl* :

for this to work some restriction [11] is needed on $T : A \to B$. If T has inverse-closed range,

9.8 $$B^{-1} \cap T(A) \subseteq T(A)^{-1} \ ,$$

then there is inclusion

9.9 $$Cl(A^{-1}) \cap T^{-1}B^{-1} \subseteq A^{-1} + T^{-1}(0) \ :$$

because

$$a \in T^{-1}B^{-1} \Longrightarrow \exists d \in A \ : \ \{1 - ad, 1 - da\} \subseteq T^{-1}(0)$$

and

$$a \in Cl(A^{-1}) \Longrightarrow \exists c \in A^{-1} \ : \ 1 - (a - c)d = e^{-1} \in A^{-1}$$

giving

$$a = e(1-(a-c)d)a = e(1-ad)a + ec(da-1) + ec \in T^{-1}(0) + T^{-1}(0) + A^{-1} \ .$$

Conversely iff T has *weakly Riesz* [5] null space

9.10 $$1 + T^{-1}(0) \subseteq Cl(A^{-1}) \ ,$$

then there is inclusion

9.11 $$A^{-1} + T^{-1}(0) \subseteq Cl(A^{-1}) \cap T^{-1}B^{-1} \ :$$

because

$$A^{-1} + T^{-1}(0) = A^{-1}(1 + T^{-1}(0)) \subseteq A^{-1}Cl(A^{-1}) \ .$$

Also necessary and sufficient for $T^{-1}(0)$ to be weakly Riesz is that it have "stable rank zero", in the sense

9.12 $$1 + T^{-1}(0) \subseteq Cl(1 + T^{-1}(0))^{-1} \ .$$

4.10 Stable range

There is a property of rings which can be thought of as a non commutative analogue of "topological dimension", in that when specialized to the ring $C(X)$ it reflects the dimension of the topological space X. Roughly, for Banach algebra, "dimension zero" says that the invertible group is (norm) dense in the whole ring. A curious reduction property for "jointly invertible" tuples intervenes in the following hybrid result: *If*

10.1 $$A^n \subseteq Cl_{right}(A^{-n}_{right})$$

then for arbitrary $(a,b) \in A^n \times A$ *there is implication*

10.2 $$(a,b) \in A^{-n-1}_{left} \subseteq A^n \times A \Longrightarrow (a - A^n b) \cap A^{-n}_{left} \neq \emptyset \ .$$

Here we are making a different definition of "closure" for the product A^n: writing for $x, y \in A^n$

10.3 $$y \cdot x = \sum_{j=1}^{n} y_j x_j$$

we declare for $K \subseteq A^n$ that $x \in Cl_{right}(K)$ iff

10.4 $$\forall\ J \in \mathrm{Finite}(A^n) \ , \ \exists\ x' \in K \ : \ 1 - (x - x') \cdot J \subseteq A^{-1} \ .$$

Indeed suppose

$$a' \cdot a + b'b = 1 \text{ with } a' \in Cl_{right}(A^{-n}_{right}) \ ,$$

so that there are $a'', a''' \in A^n$ with

$$b'b = 1 - a' \cdot a = d - a'' \cdot a \text{ with } d \in A^{-1} \ , \ a'' \cdot a''' = 1 \ :$$

then $d^{-1}a'' \cdot (a + a'''b'b) = d^{-1}(a'' \cdot a + b'b)$ giving

$$a - cb = a'''d \in A^{-n}_{left} \text{ with } c = -a'''b' \ \bullet$$

This is essentially Theorem 2.3 of [10]. When $b = 0$ it nearly gives back (5.3). For a C* algebra A the condition (10.1), its norm analogue, and (10.2) are [8] equivalent; in contrast when A is the disc algebra then (10.2) holds ([9] Theorem 1) with $n = 1$, while

10.3 $$\mathrm{cl}(A^{-1}) \neq A \ :$$

this makes it acutely interesting to know whether or not (10.1) holds.

Ara, Pedersen and Perera [1],[2] define $a \in Cl^{\sim}_{left}(K)$ to mean that for arbitrary $b \in A$ there is implication

10.4 $$(a, b) \in A^{-2}_{left} \subseteq A^2 \Longrightarrow (a - Ab) \cap K \neq \emptyset \ .$$

This is another kind of "algebraic closure", and indeed satisfies the conditions $[Ku_0]-[Ku_3]$ but not $[Ku_4]$. We believe that it admits a modification similar to ours, if we instead define, for $K \subseteq A^n$,

$$a \in Cl^{\sim}_{left}(K) \subseteq A^n \Longleftrightarrow \forall J \in \mathrm{Finite}(A) \ : (a, J) \subseteq A^{-n-1}_{left}$$

10.5 $$\Longrightarrow \exists a' \in K \ : a' \in \bigcap_{b \in J} (a - A^n b) \ .$$

With this notation the implication (10.1)$\Longrightarrow$(10.2) says

10.6 $$A \subseteq Cl_{right} A^{-n}_{right} \Longrightarrow A \subseteq Cl^{\sim}_{left} A^{-n}_{left} \ .$$

4.11 References

[1] Pere Ara, Gert Pedersen and Francesc Perera, *An infinite analogue of rings with stable rank one*, Jour. Algebra **230** (2000) 608-655.

[2] Pere Ara, Gert Pedersen and Francesc Perera, *A closure operation in rings*, Int. Jour. Math. **12** (2001) 791-812.

[3] Dragana Cvetković-Ilić and Robin Harte, *On the algebraic closure in rings*, Proc. Amer. Math. Soc. **135** (2007) 3547-3552.

[4] Dragana Cvetković-Ilić and Robin Harte, *The spectral topology in rings*, Studia Math. **200** (2010), 267-278.

[5] Dragan Djordjević, Robin Harte and Cora Stack, *On left-right consistency in rings*, Math. Proc. Royal Irish Acad. **106A** (2006) 11-17.

[6] Robin Harte, *Regular boundary elements*, Proc. Amer. Math. Soc. **99** (1987) 328-330.

[7] Robin Harte, *Invertibility and singularity*, Dekker 1988.

[8] R.H. Herman and L.N. Veserstein, *The stable range of C*-algebras*, Invent. Math. **77** (1984) 553-555.

[9] P.W. Jones, D. Marshall and T. Wolff, *Stable rank of the disc algebra*, Proc. Amer. Math. Soc. **96** (1986) 603-604.

[10] Marc Rieffel, *Dimension and stable rank in the K-theory of C*-algebras*, Proc. London Math. Soc. **46** (1983) 301-333

[11] Yifeng Xue, *A note about a theorem of R. Harte*, Filomat Nis **22** (2008) 95-98.

5

Approximation on compact totally disconnect groups

György Gát and Rodolfo Toledo [1]

Abstract: This work summarizes some statements with respect to Fourier analysis on compact totally disconnect groups, specifically topological groups formed by the complete direct product of finite groups with discrete topology and Haar measure. The systems with which we work are an arrangement of characters and normalized coordinate functions introduced by Paley for Walsh functions. We show the recent results we obtained in the study of the convergence of Fourier series, Fejér means and Cesàro means, emphasizing the differences between the commutative and noncommutative structures.

Keywords: Walsh, commutative and noncommutative Vilenkin groups, Fourier series, one and two dimensional Fejér, logarithmic means, almost everywhere convergence.

[1]College of Nyíregyháza, Inst. of Math. and Comp. Sci., Nyíregyháza, P.O.Box 166., H–4400, Hungary. emails: gatgy@nyf.hu and toledo@nyf.hu

5.1 Introduction

The theory of abstract harmonic analysis has been a relevant progress in the last decades. An increasing number of mathematicians have adopted the point of view that the most appropriate setting for the development of the theory of Fourier analysis is furnished by the class of all locally compact groups.

The structure of topological groups was extensively studied in the years 1925-1940, and the subject is far from dead even today. The study of the direct products of topological groups have been started since the beginning of the theory of topological groups. Pontryagin [41] examined very extensively the structure of countable direct products treated special cases of finite direct products. Vilenkin [1] obtained several results for the commutative cases.

The theory of Walsh series is a special case of the study of harmonic analysis on compact abelian groups if we represented them on the dyadic group. This group is the simplest but nontrivial model of the complete product of finite groups. A natural generalization on the Walsh-Paley system is the Vilenkin system introduced by Vilenkin [1] in 1947. He used the set of all characters of the complete product of arbitrary cyclic groups to obtain the commutative case. In Hungary a dyadic analysis team works leaded by Schipp having several results in this theory.

The idea of study the complete product of finite, but not-necessarily abelian groups is based on the many results obtained previously for Vilenkin groups and the results of Benke [4] in 1978. He proved that the Lipschitz class to which a function belongs can be identified by the best approximation characteristics of the function by trigonometric polynomials (representative product systems), and that functions which are easily approximated by trigonometric polynomials have absolutely convergent Fourier series. The authors of this work were the first who deal with the convergence in L^p-norm and almost everywhere of Fourier series and Fejér means with respect to representative product systems (see [26]). Starting of the classical theory of Fourier series and integrals the relative ease with which the basic concepts and theorems can be transferred to this general context in the abelian case is not valid for the non-commutative case.

In this work we denote by $\mathbb{N}, \mathbb{P}, \mathbb{C}$ the set of nonnegative, positive integers and complex numbers, respectively.

5.2 The Walsh functions

The Walsh functions form an orthonormal system which takes on only the values -1 and 1. This property makes these functions can be well used in practice, namely data transmission, filtering, image processing, and many other applications. Some enumerations of Walsh functions have been studied, however, in this work we are only dealing with the enumeration introduced by Paley. He was first to recognize that Walsh functions are products of Rademacher functions.

Let r be the function defined on the unit interval $[0, 1[$ by

$$r(x) := \begin{cases} 1 & \text{for } x \in [0, \frac{1}{2}[\\ -1 & \text{for } x \in [\frac{1}{2}, 1[. \end{cases}$$

extended to the real line by periodicity of period 1. The sequence of functions

$$r_n(x) := r(2^n x) \qquad (x \in [0, 1[,\ n \in \mathbb{N})$$

is called the *Rademacher system* on the unit interval $[0, 1[$. The Paley's enumeration is based on the fact that every positive integer n has an unique binary expansion

$$n = \sum_{k=0}^{\infty} n_k 2^k,$$

where $n_k = 0$ or $n_k = 1$ for all $k \in \mathbb{N}$. The *Walsh-Paley system* is defined by the sequence

$$\omega_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (x \in [0, 1[,\ n \in \mathbb{N}).$$

Notice that this product is always finite because $n_k = 1$ only for finitely many values of k. Moreover all of finite product of Rademacher functions can be written in this form.

The Walsh-Paley system is an orthonormal system, i.e.

$$\int_0^1 \omega_n(x)\omega_m(x)\,dx = \begin{cases} 0 & \text{for } n \neq m \\ 1 & \text{for } n = m. \end{cases}$$

5.3 The Dyadic group

Let $\mathcal{Z}_2$ be the discrete cyclic group of order 2, i.e. the set $\{0,1\}$ with the discrete topology and modulo 2 addition. The *Dyadic group* G is the compact group formed by the complete direct product of $\mathcal{Z}_2$ with the product of the topologies and operations $(+)$, say

$$G := \mathcal{Z}_2 \times \mathcal{Z}_2 \times \dots$$

Thus any element x of G can be represented by a sequence $x = (x_0,\, x_1, \dots)$ where $x_k = 0$ or $x_k = 1$ for all $k \in \mathbb{N}$. G is a compact topological group where the sets

$$I_n(x) := \{y \in G\colon y_i = x_i, \text{ for } 0 \leq i < n\}$$

are a countable base of the topology. We call these sets the *dyadic intervals* of G.

In order to obtain the normalized Haar measure on G we give first a measure on $\mathcal{Z}_2$ assigning each singleton the measure $\frac{1}{2}$. The product measure on G is the founded Haar measure. We denote it by μ. Measurable functions on G whose p-th power are integrable, play an important role in approximation. For $1 < p < \infty$ let $L^p(G)$ represent the set of this functions which is a Banach space with norm

$$\|f\|_p := \left(\int_G |f|^p\,d\mu\right)^{\frac{1}{p}}.$$

Since the measure μ is finite the relation

$$L^q(G) \subset L^p(G) \subset L^1(G) \qquad (1 < p < q < \infty)$$

holds. For this reason the most extensive set of functions on G we consider is just $L^1(G)$. Similarly $L^\infty(G)$ represent the set of all measurable functions f such that

$$\|f\|_\infty := \inf\{y \in \mathbb{R} : |f(x)| \leq y \text{ for a.e. } x \in G\}$$

is finite.

The characters of G are the finite many products of the characters of $\mathcal{Z}_2$, i.e. the finite many products of the functions

$$\varphi_k(x) := (-1)^{x_k} \qquad (x_k \in \{0, 1\}).$$

We enumerate the characters of G in the Paley's sense, so define

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x) \qquad (x \in G,\, n \in \mathbb{N}),$$

where $(n_0,\, n_1, \ldots)$ is the binary expansion of n. Characters are continuous complex - valued maps on G which satisfy

$$\psi(xy) = \psi(x)\psi(y) \qquad (x, y \in G).$$

and

$$|\psi(x)| = 1 \qquad (x \in G).$$

From these properties we can obtain:

Theorem 5.3.1 *The system ψ is an orthonormal and complete system on $L^2(G)$.*

The Dyadic group can be identified in the interval $[0, 1[$ in such a way that the system of characters φ on G correspond to the Walsh-Paley system ω on $[0, 1[$. For this reason we also call the system φ the Walsh-Paley system on G. The representation of G in the interval $[0, 1]$ is constructed as follows

The topology of G is metrizable. Moreover, the metric we concerned is induced by the norm

$$|x| := \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \qquad (x \in G).$$

The proceeded metric $d(x,y) := |x+y|$ induces the topology of G and $0 \leq |x| \leq 1$ for all $x \in G$. On the other hand, any $x \in [0,1[$ can be written

$$x := \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \qquad (x_k = 0 \text{ or } x_k = 1),$$

but there are numbers with two expressions of this form. They are all numbers in the set

$$\mathbb{Q} := \left\{ \frac{p}{2^n} : 0 \leq p < 2^n,\ n, p \in \mathbb{N} \right\}$$

called *dyadic rational numbers*. The other numbers have only one expression. The dyadic rational numbers have an expression terminates in 0's and other terminates in 1's. We choose the first one to make an unique relation for all numbers in the interval $[0,1[$ with their expression, named the *dyadic expansion* of the number. In this manner we assign to a number in the interval $[0,1]$ having an dyadic expansion $(x_0, x_1, \dots)$ an element of G with expansion $(x_0, x_1, \dots)$. We denote this relation by ρ called *Fine's map*. Thus

$$\omega_n := \psi_n \circ \rho \qquad (n \in \mathbb{N}).$$

and

$$\psi_n(x) := \omega_n(|x|) \qquad (x \in G \setminus \mathbb{Q},\ n \in \mathbb{N}).$$

The following theorem show the relation between the Haar integration on G and the Lebesgue integration on the interval $[0,1]$.

Theorem 5.3.2 *Let ρ denote the Fine's map.*

(a) If f is Haar measurable on G then $f \circ \rho$ is Lebesgue measurable on $[0,1[$*. Conversely, if g is Lebesgue measurable on* $[0,1[$ *and*

$$f(x) := g(|x|) \qquad (x \in G) \tag{5.1}$$

then f is Haar measurable on G.

(b) If f is Haar integrable on G then $f \circ \rho$ is Lebesgue integrable and

$$\int_G f\, d\mu = \int_0^1 (f \circ \rho)(x)\, dx.$$

Conversely, if g is Lebesgue integrable and f is defined by (5.1) then f is Haar integrable on G and

$$\int_0^1 g(x)\, dx = \int_G f\, d\mu.$$

We summarize now the results with respect to the convergence of Walsh-Fourier series. For $f \in L^1(G)$ we define the *Fourier coefficients* by

$$\widehat{f}_k := \int_G f \overline{\psi}_k d\mu \qquad (k \in \mathbb{N}),$$

and *the n-th partial sums of Walsh-Fourier series* by

$$S_n f(x) := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k(x) \qquad (x \in G,\, n \in \mathbb{P}).$$

Theorem 5.3.3 *If $1 < p < \infty$ and $f \in L^p(G)$ then the n-th partial sums of Walsh-Fourier series $S_n f$ converge to f a.e. and in L^p norm.*

The above theorem is not true for $p = 1$. The *Dirichlet kernels* are defined as follows:

$$D_n(x) := \sum_{k=0}^{n-1} \psi_k(x) \qquad (n \in \mathbb{P}).$$

It is easy to see that

$$S_n f(x) = \int_G f(y) D_n(x + y) d\mu(y).$$

This mutates the importance of the Dirichlet kernels in the study of the convergence of Fourier series.

Lemma 5.3.1 (Paley lemma) *Let e be the identity of G and $I_n := I_n(e)$. If $n \in \mathbb{N}$ and $x \in G$, then*

$$D_{2^n}(x) = \begin{cases} 2^n & \text{for } x \in I_n, \\ 0 & \text{for } x \notin I_n \end{cases}$$

The Paley lemma is used to prove that the $S_{2^n} f$ partial sequence of Fourier sums converge to f a.e. and in L^p-norm, if $f \in L^p(G)$ and $1 \le p < \infty$. Indeed, the

$$S_{2^n} f(x) = \int_G f(y) D_{2^n}(x+y) d\mu(y) = \frac{1}{\mu(I_n(x))} \int_{I_n(x)} f d\mu$$

operator is the conditional expectation with respect to the σ-algebra generated by the sets $I_n(x)$, $x \in G$. Thus, the following statement is a consequence of the martingale convergence theorem (see [38]).

5.4 Vilenkin systems

Vilenkin systems are generalizations of the Walsh-Paley system which were introduced by Vilenkin [1] in 1947. These are systems composed of the set of all characters of the complete product of arbitrary cyclic groups. In this regard we introduce the following notation.

Let $m := (m_k,\ k \in \mathbb{N})$ be a sequence of positive integers such that $m_k \ge 2$ and $\mathcal{Z}_k$ the cyclic group with order m_k, $(k \in \mathbb{N})$. Suppose that each group has discrete topology and normalized Haar measure μ_k. A *Vilenkin group G* is defined by the compact group formed by the complete direct product of $\mathcal{Z}_k$ with the product of the topologies, operations and measures (μ). Thus each $x \in G$ consist of sequences $x := (x_0, x_1, ...)$, where $0 \le x_k < m_k$, $(k \in \mathbb{N})$. We call this sequence the *expansion of x*.

We can generalize the concept of Rademacher functions for cyclic groups with order $m_k > 2$ as follows

$$\varphi_k^s(x) = \exp(2\pi\imath sx/m_k) \qquad (s \in \{0, \ldots m_k - 1\},\ x \in \mathcal{Z}_{m_k},\ \imath^2 = -1). \quad (5.2)$$

Generalized Rademacher functions are the characters of the cyclic group. Thus the characters of the product group G are the set of all finite numbers of functions given by (5.2). The construction of the system of characters is similar to the Paley's method, but we work with the sequence

$$M_0 := 1, \qquad M_{k+1} := m_k M_k \qquad (k \in \mathbb{N})$$

instead the powers of 2. Indeed every $n \in \mathbb{N}$ can be uniquely expressed as

$$n = \sum_{k=0}^{\infty} n_k M_k, \qquad (0 \le n_k < m_k,\ n_k \in \mathbb{N}).$$

This allows us to say that the sequence $(n_0, n_1, \dots)$ is the *expansion of* n *with respect to* m. With the expansion of n and x we define the system ψ by

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x \in G),$$

and it is called a *Vilenkin systems*. In this case he property of characters also secures:

Theorem 5.4.1 *The system ψ is an orthonormal and complete system on $L^2(G)$.*

For an integrable complex function f defined in G we define Vilenkin-Fourier coefficients and partial sums by

$$\widehat{f}_k := \int_{G_m} f\overline{\psi}_k \, d\mu \quad (k \in \mathbb{N}), \qquad S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k \quad (n \in \mathbb{N}). \tag{5.3}$$

In terms of the convergence of Vilenkin-Fourier series we have to see the sequence m from which we obtain the Vilenkin group G. If it is a bounded sequence, we called G a *bounded group*. For bounded Vilenkin groups we can use the method applied for the dyadic case with relative similitude. Thus we obtain:

Theorem 5.4.2 *Let G be a bounded Vilenkin group. If $1 < p < \infty$ and $f \in L^p(G)$ then the n-th partial sums of Walsh-Fourier series $S_n f$ converge to f a.e. and in L^p norm.*

The unbounded cases are more difficult to treat. However Young [60], Schipp [45] and Simon [48] can proved the convergence in L^p norm for an arbitrary Vilenkin group.

Theorem 5.4.3 (Young, Schipp and Simon) *Let G be a Vilenkin group. If $1 < p < \infty$ and $f \in L^p(G)$ then the n-th partial sums of Vilenkin-Fourier series $S_n f$ converge to f in L^p norm.*

The almost everywhere convergence is not yet proved for unbounded Vilenkin groups, i.e. the problem corresponding to the Theorem of Carleson for the trigonometric system, is open. It makes more relevant the fact that for all $f \in L^p(G)$ and $1 \le p < \infty$ there exists a partial sequence of Fourier sums which converges to f a.e. Indeed, the partial sequence $S_{M_n} f$ is also the conditional expectation with respect to the σ-algebra generated by the sets $I_n(x)$, $x \in G$, so we can applied the martingale convergence theorem.

5.5 Results and open problems

First, we would like to talk about almost everywhere summability of Walsh series of one variable integrable functions. The n-th $(C, 1)$ mean, the n-th Riesz's logarithmic mean of $f \in L^1(G)$:

$$\sigma_n f(y) := \frac{1}{n} \sum_{k=0}^{n-1} S_k f(y),$$

$$R_n f(y) := \frac{1}{\log n} \sum_{k=1}^{n} \frac{S_k f(y)}{k}.$$

Let have a look for the situation with the (C, α) means. What are they? Later on, we are going to introduce them in details at section 9. Now, briefly let $A_n^\alpha := \frac{(1+\alpha)\dots(n+\alpha)}{n!}$, where $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ $(-\alpha \notin \mathbb{N})$. The n-th (C, α) mean of the function $f \in L^1(G)$:

$$\sigma_{n+1}^\alpha f = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f.$$

It is of main interest in the theory of Fourier series that how to reconstruct the function from the partial sums of its Fourier series. Just to mention two examples: Billard proved [5] the theorem of Carleson for the Walsh-Paley system, that is, for each function in L^2 we have the almost everywhere convergence $S_n f \to f$. Fine [8] proved every Walsh-Fourier series (in the Walsh case $m_j = 2$ for all $j \in \mathbb{N}$) is a.e. (C, α) summable for $\alpha > 0$. His argument is an adaptation of the older trigonometric analogue due to Marcinkiewicz [36]. In 1975 Schipp [45] - with respect to the Walsh case - gave a simpler proof for the case $\alpha = 1$, i.e. $\sigma_n f \to f$ a.e. $(f \in L^1(G))$. He proved that $\sigma^* := \sup |\sigma_n|$ is of weak type (L^1, L^1). This means that $\sup_{\lambda>0} \mu(\sigma^* f > \lambda) \leq C\|f\|_1$. This led to a new technique to prove the a.e. convergence of Fejér means and also Cesàro (C, α) means. The theorem of $(C, 1)$ summability is generalized to the p-series fields ($m_j = p$ for all $j \in \mathbb{N}$) by Taibleson [49], and later to bounded Vilenkin systems by Pál and Simon [40]. The result of Fine [8] and Schipp [45] on bounded Vilenkin groups is due to Weisz [57]. In other words, with respect to the Walsh or bounded Vilenkin systems the maximal convergence space of the (C, α) means is the L^1 Lebesgue space. That is, the largest possible.

Now, what about the Vilenkin groups with unbounded generating sequences? This is quite a different story. The methods known in the trigonometric or in the Walsh, bounded Vilenkin case are not powerful enough. One of the main problems is that the proofs on the bounded Vilenkin groups (or in the trigonometric case) heavily use the fact that the L^1 norm of the Fejér kernels are uniformly bounded. This is not the case if the group G (that is the generating sequence m) is an unbounded one [42]. From this it follows that the original theorem of Fejér does not hold on unbounded Vilenkin groups. Namely, Price proved [42] that for an arbitrary sequence m ($\sup_n m_n = \infty$) and $a \in G$ there exists a function f continuous on G_m and $\sigma_n f(a)$ does not converge to $f(a)$. Moreover, he proved [42] that if $\frac{\log m_n}{M_n} \to \infty$, then there exists a function f continuous on G whose Fourier series are not $(C,1)$ summable on a set $S \subset G$ which is non-denumerable.

Moreover, the result of Price also implies that for each unbounded Vilenkin group G one can give an integrable function $f \in L^1(G)$ such that even the special subsequence of the Fejér means $\sigma_{M_n} f$ do not converge to the function in the Lebesgue norm L^1.

On the other hand, norm convergence of the full partial sums for $L^p, p > 1$, is known for the unbounded case. This result is proven by Schipp - as it can be read above. This trivially implies the norm convergence $\sigma_n f \to f$ for all $f \in L^p$, where $1 < p < \infty$. But what positive can be said with respect to the L^1 case?

The concept of Nörlund logarithmic means is as follows

$$t_n f := \frac{1}{\log n} \sum_{k=1}^{n-1} \frac{S_k f}{n-k}.$$

For further information with respect to Nörlund logarithmic means on Walsh-Paley systems see some papers of Gát and Goginava and Tkebuchava [24,23,25]. In their paper Gát and Goginava [24] proved (for Walsh-Paley system), that there exists an $f \in L^1$ such that

$$\|t_n f - f\|_1 \not\to 0.$$

On the other hand, Blahota and Gát [6] proved that the Nörlund logarithmic means have better approximation properties on some unbounded Vilenkin groups, than the Fejér means. Namely:

Theorem 5.5.1 *If $f \in L^1$ and*

$$\limsup_{n \in \mathbb{N}} \frac{\sum_{k=0}^{n-1} \log^2 m_k}{\log M_n} < \infty,$$

then

$$\|t_{M_n} f - f\|_1 \to 0.$$

In the case $f \in C$ the convergence holds in the supremum norm. This means that in the case of some unbounded Vilenkin groups the behavior of the Nörlund means t_{M_n} means is better than the behavior of the Fejér means σ_{M_n}.

On the other hand, this can not be said in general, that is for the means t_n. That is, Blahota and Gát proved [6]:

If $\log m_n = O(n^\delta)$ for some $0 < \delta < 1/2$, then there exists an $f \in L^1$ such that

$$\|t_n f - f\|_1 \not\to 0.$$

It is surprising that the behavior of the Nörlund logarithmic means is worse than the behavior of the Fejér means in the Walsh-Paley or in the bounded Vilenkin case, but the situation changes on a class of unbounded Vilenkin groups. For the time being it is an open question that it is possible to give an unbounded generating sequence m such that we would have the norm convergence $\|t_n f - f\|_1 \to 0$ for all integrable function f.

We already have written about the behavior of the Nörlund logarithmic means. Another weighted mean of the partial sums of the Fourier series is the Riesz logarithmic means, which seems to be very similar to the Nörlund ones (recall its definition):

$$\frac{1}{\log n} \sum_{k=1}^{n-1} \frac{S_k f}{k}.$$

Problem. It is easy to see in the trigonometric, Walsh and bounded Vilenkin case, that for each integrable function f the Riesz logarithmic means converge to f both in norm and a.e. This is a trivial consequence of the nice properties of the Fejér means and the Abel transformation. On the other hand, if we investigate these means on unbounded Vilenkin groups then the situation is different. Namely, for the time being there is no result with respect to convergence of these Riesz means of integrable functions.

Now, turn back to the Fejér means. Nurpeisov gave [39] a necessary and sufficient condition of the uniform convergence of the Fejér means $\sigma_{M_n} f$ of continuous functions on unbounded Vilenkin groups. Namely, define the uniform modulus of continuity as

$$\omega_n(f) := \sup_{h\in I_n(0), x\in G} |f(x+h) - f(x)|.$$

Let ω be a real sequence with property $\omega_n \searrow 0$.We say that f belongs to the Hölder class H^ω if $\omega_n(f) \leq \omega_n$ for all $n \in \mathbb{N}$. Nurpeisov proved [39]: A necessary and sufficient condition that the means $\sigma_{M_n} f$ of the Fourier series of the continuous function f converge uniformly to f on an unbounded Vilenkin group for all f belonging to the Hölder class ω is that

$$\omega_{n-1}(f)\log(m_n) = o(1).$$

Since the uniform modulus of continuity can be any nonincreasing real sequence which converges to zero (for the proof see [43], [9]), then as a consequence of this it is possible to give a sequence m increasing enough fast, and a function even in the Lipschitz class Lip(1), such that the M_nth Fejér means do not converge to the function uniformly.

So, it seems that it is impossible to give a (Hölder) function class such that the uniform convergence of the Fejér means would hold for all functions in this class if there is no condition on sequence m at all.

It also seems that some difference could occur in the case of Nörlund logarithmic means. For the time being there is no result is known with respect to this issue.

Concerning the a.e. convergence and Fejér means on unbounded Vilenkin groups we can say a bit more. Namely, in 1999 one of the authors [11] proved:

Theorem 5.5.2 *If $f \in L^p(G_m)$, where $p > 1$, then $\sigma_n f \to f$ almost everywhere.*

This was the very first "positive" result with respect to the a.e. convergence of the Fejér means of functions on unbounded Vilenkin groups. One might say that this result is an easy consequence of the result of Carleson, that is the a.e. convergence $S_n f \to f$ for functions $f \in L^p(G)$, where $p > 1$.

The "only problem" is that to prove this a.e. convergence result of the partial sums is the one of the greatest open problems in the theory of Fourier analysis on Vilenkin groups.

However, it is possible to step further in the direction of space $L^1(G)$. In 2001 Simon proved [47] the following theorem with respect to the Fejér means of L^1 functions. A sequence m is said to be strong quasi-bounded if

$$\frac{1}{M_{n+1}} \sum_{j=0}^{n-1} M_{j+1} < C \log m_n.$$

Then every bounded m is quasi-bounded, and there are also some unbounded ones. Let m be strong quasi-bounded. Then for all $f \in L^1(G)$

$$\sigma_{M_n} f(x) - f(x) = o(\max(\log m_0, \ldots, \log m_{n-1})).$$

Later, in 2003, Gát improved [14] this result, and gave a partial answer for the L^1 case. He discussed this partial sequence of the sequence of the Fejér means. Namely,

Theorem 5.5.3 *if $f \in L^1(G_m)$, then ([14]) $\sigma_{M_n} f \to f$ almost everywhere, where m is any sequence.*

Remark. Recall that we have told that it is possible to give integrable function $f \in L^1(G)$ such that even the special subsequence of the Fejér means $\sigma_{M_n} f$ do not converge to the function in the Lebesgue norm L^1 and in spite of this fact the a.e. convergence does hold. In our opinion this is a very interesting property of the unbounded Vilenkin systems.

Problem. In our opinion, it is highly likely that the methods of the papers [11,14] can be applied and improved in order to prove the a.e. relation $\sigma_n f \to f$ for all $f \in L \log^+ L$ and m. Anyway, it is not an easy task...

With respect to another class of unbounded Vilenkin groups Gát proved the original Lebesgue theorem. This class is called "rarely unbounded". What does it mean?

If there exists a constant C and $L \in \mathbb{N}$ such that for all $i, j \in \mathbb{N}$ we have

$$\frac{\min(m_i, m_{i+j})}{(m_{i+1} \cdot \dots \cdot m_{i+j-1})^L} \leq C$$

(the empty product is defined to be 1, and the constant C may depend on the sequence m - of course), then we call the Vilenkin group G a rarely unbounded Vilenkin group. Every bounded Vilenkin group is a rarely unbounded Vilenkin group. Unfortunately, not all unbounded ones are rarely unbounded, since for instance the rarely unboundedness implies the inequality $\min(m_i, m_{i+1}) \leq C$. So, e.g. if (m_n) tends to plus infinity, then G is not rarely unbounded. On the other hand, there are many unbounded Vilenkin groups, which are rarely unbounded ones.

In paper [18] one can find:

Theorem 5.5.4 *Let G be a rarely unbounded Vilenkin group. Then the operator σ^* is of weak type $(1,1)$.*

A straightforward consequence of Theorem 5.5.4 is the proof of the Fejér-Lebesgue theorem on rarely unbounded Vilenkin groups. That is,

Theorem 5.5.5 *Let G be a rarely unbounded Vilenkin group and $f \in L^1(G)$. Then we have the a.e. relation $\sigma_n f \to f$.*

It is also interesting to add that the concept of rarely unbounded Vilenkin groups is natural in the point of view of the Carleson's theorem. Since it can be proved that if the theorem of Carleson holds on every rarely unbounded Vilenkin group, then it also holds on every Vilenkin groups.

Problem. Nothing has done on unbounded Vilenkin groups with respect to (C, α) $(0 < \alpha < 1)$ or Riesz summability. In our opinion it would be possible to investigate these means by the help of methods in the papers of Gát regarding Fejér means on unbounded Vilenkin groups. First, we would suggest to try with the following summation operators:

$$\frac{1}{\log M_n} \sum_{k=1}^{M_n} \frac{S_k f}{k}, \quad \frac{1}{A_{M_n}^{\alpha}} \sum_{k=0}^{M_n} A_{n-k}^{\alpha-1} S_k f.$$

It is also of prior interested that what can be said - with respect to this reconstruction issue - if we have only a subsequence of the partial sums. In 1936 Zalcwasser [62] asked how "rare" can be the sequence of integers $a(n)$ such that

$$\frac{1}{N}\sum_{n=1}^{N} S_{a(n)}f \to f. \tag{5.4}$$

This problem with respect to the trigonometric system was completely solved for continuous functions (uniform convergence) in [44], [61], [2] and [7]. That is, if the sequence a is convex, then the condition

$$\sup_n n^{-1/2} \log a(n) < +\infty$$

is necessary and sufficient for the uniform convergence for every continuous function. For the time being, this issue with respect to the Walsh-Paley system has not been solved. Only, a sufficient condition is known, which is the same as in the trigonometric case. The paper about this is written by Glukhov [28]. See the more dimensional case also by Glukhov [29].

With respect to convergence almost everywhere, and integrable functions the situation is more complicated. Belinksky proved [3] for the trigonometric system the existence of a sequence $a(n) \sim \exp(\sqrt[3]{k})$ such that the relation (5.4) holds a.e. for every integrable function. In this paper Belinksky also conjectured that if the sequence a is convex, then the condition $\sup_n n^{-1/2} \log a(n) < +\infty$ is necessary and sufficient again. So, that would be the answer for the problem of Zalcwasser [62] in this point of view (trigonometric system, a.e. convergence and L^1 functions). One of the authors of this paper proved [19] that this is not the case for the Walsh-Paley system. See below Theorem 5.5.6. On the other hand, this difference between the Walsh-Paley and the trigonometric system is not so surprising. Because of the following. Let $v(n) := \sum_{i=0}^{\infty} |n_i - n_{i+1}|, (n = \sum_{i=0}^{\infty} n_i 2^i)$ be the variation of the natural number n expanded in the number system based 2. It is a well-known result in the literature that for each sequence a tending strictly monotone increasing to plus infinity with the property $\sup_n v(a(n)) < +\infty$ we have the a.e. convergence $S_{a(n)}f \to f$ for all integrable function f. Is it also a necessary condition? This question of Balashov was answered by Konyagin [34] in the negative. He gave an example. That is, a sequence a with property $\sup_n v(a(n)) = +\infty$ and he proved that $S_{a(n)}f \to f$ a.e. for all integrable function f.

In paper [19] it is proved (see Theorem 5.5.6) that for each lacunary sequence a (that is $a(n+1)/a(n) \geq q > 1$) and each integrable function f the relation (5.4) holds a.e. This may also be interesting in the following point of view. If the sequence a is lacunary, then the a.e. relation $S_{a(n)}f \to f$ holds for all functions f in the Hardy space H. The trigonometric and the Walsh-Paley case can be found in [65] (trigonometric case) and [35] (Walsh-Paley case). But, the space H is a proper subspace of L^1. Therefore, it is of interest to investigate relation (5.4) for L^1 functions and lacunary sequence a.

In that paper - using the method of the proof of Theorem 5.5.6 one can find (Theorem 5.5.7) that for any convex sequence a (with $a(+\infty) = +\infty$ - of course) and for each integrable function the Riesz's logarithmic means of the function converges to the function almost everywhere. That is, the Riesz's logarithmic summability method can reconstruct the corresponding integrable function from any (convex) subsequence of the partial sums in the Walsh-Paley situation. For the time being there is no result known with respect to a.e. convergence of logarithmic means of subsequences of partial sums, neither in the trigonometric nor in the Vilenkin case.

The following a.e. convergence theorems with respect to the Fejér and logarithmic means of subsequences of the partial sums of the Walsh-Fourier series of integrable functions are proved by Gát [19].

Theorem 5.5.6 *Let $a : \mathbb{N} \to \mathbb{N}$ be a sequence with property $\frac{a(n+1)}{a(n)} \geq q > 1\,(n \in \mathbb{N})$. Then for all integrable function $f \in L^1(Q)$ we have the a.e. relation*

$$\frac{1}{N}\sum_{n=1}^{N} S_{a(n)}f \to f.$$

Theorem 5.5.7 *Let $a : \mathbb{N} \to \mathbb{N}$ be a convex sequence with property $a(+\infty) = +\infty$. Then for each integrable function f we have the a.e. relation*

$$\frac{1}{\log N} \sum_{n=1}^{N} \frac{S_{a(n)} f}{n} \to f.$$

Problems and remarks. Here we remark, that these two theorems have no antecedents in the theory of trigonometric system. Therefore it is of prior interest to investigate them. We also mention that there is no result with respect to this issue concerning the Vilenkin systems (even in the more simple bounded case). Below, we introduce some necessary preliminaries and notations related the investigation of the two dimensional Walsh and Vilenkin systems and also deliver some results and problems. There is no result at all concerning the two (or more) dimensional case even for the Walsh system with respect to these a.e. convergence results above.

Problem. It is highly likely that one can investigate the problem of a.e. convergence of (C, α) means of subsequence of the partial sums of Walsh-Fourier (or Vilenkin-Fourier (bounded case)) series. That is, to discuss the behavior of $\frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_{a(k)} f$. For the time being, nothing has done into this direction yet.

What can be said in the two (more) dimensional situation? This is quite a different story. Define the two-dimensional Walsh-Paley functions in the following way:

$$\psi_n(x) := \omega_{n_1}(x^1) \psi_{n_2}(x^2),$$

where $n = (n_1, n_2) \in \mathbb{N}^2$, $x = (x^1, x^2) \in G^2$. Let f be an integrable function. Its Fourier coefficients, partial sums of its Fourier series:

$$\widehat{f}_n := \int_{G^2} f(x) \psi_n(x) dx,$$

$$S_{n_1, n_2} f := \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \widehat{f}_{k_1, k_2} \psi_{k_1, k_2}.$$

Moreover, the two-dimensional Fejér or $(C,1)$ means of the function $f \in L^1(G^2)$:

$$\sigma_{n_1,n_2} f := \frac{1}{n_1 n_2} \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} S_{k_1,k_2} f \quad (n \in \mathbb{P}^2).$$

In 1931 Marczinkiewicz and Zygmund proved for the two-dimensional trigonometric system [36], and in 1992 Móricz, Schipp and Wade verified for the two-dimensional Walsh-Paley system, that for every $f \in L \log^+ L(G^2)$

$$\sigma_{n_1,n_2} f \to f$$

a.e. as $\min\{n_1, n_2\} \to \infty$, that is, in the Pringsheim sense.

Since $L \log^+ L(G^2)$ is a proper subspace of $L^1(G^2)$, then it would be interesting to "enlarge" the convergence space, if possible. In 2000 Gát proved [12] (in the Walsh case), that it is impossible. That is, for each measurable function $\delta : [0, +\infty) \to [0, +\infty)$, $\delta(\infty) = 0$, (that is vanishing at plus infinity) there exists a

$$f \in L \log^+ L\delta(L) \quad \text{such that} \quad \sigma_{n_1,n_2} f \not\to f$$

a.e. (in the Pringsheim sense).

However, what "positive" can be said for the functions in $L^1(G^2)$ as if the a.e. convergence of the two-dimensional Fejér means in the Pringsheim sense can not be said? That could be the so called restricted convergence. For the two-dimensional trigonometric system Marcinkiewicz and Zygmund proved [33] in 1939, that

$$\sigma_{n_1,n_2} f \to f$$

a.e. for every $f \in L^1(G^2)$ as if $\min\{n_1, n_2\} \to \infty$, provided that

$$2^{-\alpha} \le \frac{n_1}{n_2} \le 2^{\alpha}$$

for some $\alpha \ge 0$. In other words, the set of admissible indices (n_1, n_2) remains in some cone. This theorem for the two-dimensional Walsh-Paley system was verified by Móricz, Schipp and Wade in 1992 in the case when n_1, n_2 both are powers of two.

$$\sigma_{2^{n_1},2^{n_2}} f \to f$$

a.e. for every $f \in L^1(G^2)$ as if $\min\{n_1, n_2\} \to \infty$, provided that $|n_1 - n_2| \le \alpha$ for some $\alpha \ge 0$.

The proof of the Marcinkiewicz-Zygmund theorem [33] (with respect to the Walsh-Paley system) for arbitrary set of indices remaining in some cone is due to Gát and Weisz [10] and [15], separately in 1996.

It is an interesting question that is it possible to weaken somehow the "cone restriction" in a way that a.e. convergence remains for each function in L^1. Maybe for some "interim space" if not for space L^1. The answer is negative both in the point of view of space and in the point of view of restriction. Namely, in 2001 Gát proved [13] the theorem below:

Let $\delta : [0, +\infty) \to [0, +\infty)$ measurable, $\delta(+\infty) = 0$ and let $w : \mathbb{N} \to [1, +\infty)$ be an arbitrary increasing function such that

$$\sup_{x \in \mathbb{N}} w(x) = +\infty.$$

Moreover, $\vee n := \max(n_1, n_2)$, $\wedge n := \min(n_1, n_2)$. The, there exists a function $f \in L \log^+ L\delta(L)$ such that

$$\sigma_{n_1,n_2} f \not\to f$$

a.e. as $\wedge n \to \infty$ such that the restriction condition $\frac{\vee n}{\wedge n} \le w(\wedge n)$ is also fulfilled. That is there is no "interim" space. Either we have space $L \log^+ L$ and "no restriction at all", or the "cone restriction" and then the maximal convergence space is L^1. As a consequence of this we have that

$$\sigma_{n_1,n_2} f \to f$$

a.e. for each $f \in L(G^2)$ as $\min\{n_1, n_2\} \to \infty$, provided that

$$\frac{\vee n}{\wedge n} \le w(\wedge n)$$

if and only if

$$\sup w(x) < \infty.$$

Another question. What is the situation with the (C, α) summation of 2-dimensional Walsh-Fourier series? What is this?

$$\sigma^{\alpha}_{n_1+1,n_2+1} f = \frac{1}{A^{\alpha}_{n_1} A^{\alpha}_{n_2}} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} A^{\alpha-1}_{n_1-k_1} A^{\alpha-1}_{n_2-k_2} S_{k_1,k_2} f.$$

In 1999 Weisz proved [57], that

$$\sigma^{\alpha}_{n_1,n_2} f \to f$$

a.e. as $\min\{n_1, n_2\} \to \infty$ for each $f \in L \log^+ L(G^2)$ and $\alpha > 0$.

The question is that is it possible to give a "larger" convergence space for the (C, α) summability method $(\alpha > 0)$? Is there such an α? If $\alpha \leq 1$, then not. Because for the $(C, 1)$ method one can not give such a "larger" space.

Problems. What is the situation with the (C, α) methods, for $\alpha > 1$? This question - in our opinion it could be investigated by the method of [12]. Also would be interesting to investigate in the point of view above, the behavior of the two dimensional Riesz logarithmic means. Is its maximal convergence space larger then $L \log^+ L$ - as in the case of the Fejér summation? Or is it the same?

What about this issue in the point of view of Vilenkin systems? As, it has written, Gát proved [12] that the maximal convergence space of the two dimensional Fejér means $\sigma_{n,k}$ is the space $L \log^+ L$. This theorem of Gát is generalized on bounded two dimensional Vilenkin groups [17]. This divergence result has not been proved for unbounded two dimensional Vilenkin groups yet. It is interesting in the following point view. It is very usual that to prove some divergence result with respect to unbounded Vilenkin systems is easier or less complicated than in the case of bounded Vilenkin systems or in the Walsh-Paley setting. That is, - when we try to determine the maximal convergence space of the two dimensional Vilenkin-Fejér means in the Pringsheim setting, - we have a little bit unusual situation.

What "positive" can be said in the two-dimensional case with respect to unbounded Vilenkin systems? In 1997 Wade proved [54] the following. Let

$$\beta_{k,j} := \max\{m_0, \dots, m_{k-1}, \tilde{m}_0, \dots, \tilde{m}_{j-1}\}.$$

The sequence m is called δ-quasi bounded, $0 \le \delta < 1$, if the sums

$$\sum_{j=0}^{n-1} m_j/(m_{j+1}\dots m_n)^\delta$$

are (uniformly) bounded. Let the generating sequences $m, \tilde{m}$ be δ-quasi bounded. Then for all (at this point denote by G_m and $G_{\tilde{m}}$ the two Vilenkin groups) $f \in L^1(G_m \times G_{\tilde{m}})$ we have

$$\sigma_{M_n,\tilde{M}_k} f(x) - f(x) = o(\beta_{n,k}\beta^{2r}_{n+r,k+r}),$$

as $n, k \to \infty$, provided that $|n-k| < \alpha$, where $\alpha, r \in \mathbb{N}$ are some constants for almost every $x \in G_m \times G_{\tilde{m}}$.

On the other hand, there was nothing concerning the almost everywhere convergence (or divergence) before the following result of the author. In [16] he proved:

Theorem 5.5.8 *Let $f \in (L\log^+ L)(G_m \times G_{\tilde{m}})$. Then we have $\sigma_{M_{n_1},\tilde{M}_{n_2}} f \to f$ almost everywhere, where $\min\{n_1, n_2\} \to \infty$ provided that the distance of the indices is bounded, that is, $|n_1 - n_2| < \alpha$ for some fixed constant $\alpha > 0$.*

Here it is necessary to emphasize that in this paper $m, \tilde{m}$ can be any sequences.

It also seems to be interesting to discuss the almost everywhere convergence of Marcinkiewicz means $\frac{1}{n}\sum_{j=0}^{n-1} S_{j,j}f$ of integrable functions on two-dimensional unbounded Vilenkin groups. Although, this mean is defined for two-variable functions, in the view of almost everywhere convergence there are similarities with the one-dimensional case. For the trigonometric, Walsh-Paley, and bounded Vilenkin case see the papers of Zhizhiasvili, Weisz and

Gát [63,56,15]. With respect to the Walsh case see also the papers of Goginava [30,31]. Some results can also be found in [58,59].

Problems. It is highly likely that by the application of the method of the proof of the a.e. relation $\sigma_{M_n} f \to f$ (on unbounded Vilenkin groups), it would be possible to prove the a.e. relation

$$\frac{1}{M_n} \sum_{j=0}^{M_n-1} S_{j,j} f \to f$$

with respect to unbounded Vilenkin systems for every integrable f. What about the (C, α) or logarithmic means of quadratic partial sums? What about the situation if we take only a subsequence of the quadratic partial sums?

Recall the definition of the n-th Nörlund logarithmic mean of the integrable function f:

$$t_n f := \frac{1}{\log n} \sum_{k=1}^{n-1} \frac{S_k f}{n-k}.$$

In 1992 Móricz and Siddigi [37] raised the question about the validity of the norm convergence $\|t_n f - f\|_1 \to 0$ for each $f \in L^1$. This holds for the Riesz logarithmic means. That is for, $R_n f := \frac{1}{l_n} \sum_{k=1}^{n} \frac{S_k f}{k}$ we have $R_n f \to f$ in norm and also a.e. The answer for the question of Móricz is negative. In 2006 Gát and Goginava proved [20] that the maximal convergence space of the L^1-norm convergence is $L \log^+ L$. Namely, let $\delta : [0, +\infty) \to [0, +\infty)$ be measurable and $\delta(+\infty) = 0$. Then there exists a $h \in L \log^+ L \delta(L)(G)$, such that $t_n h \not\to h$ in L^1-norm. What can be said in the two-dimensional case? In 2004 Gát and Goginava [21] introduced the Nörlund logarithmic mean operator of the square partial sums of the two-dimensional Fourier series:

$$t_n f(x, y) = \frac{1}{\log n} \sum_{i=1}^{n-1} \frac{S_{i,i} f(x, y)}{n-i},$$

and the Nörlund logarithmic mean operator of the two-dimensional partial sums of the two-dimensional Fourier series:

$$t_{n,m}f(x,y) = \frac{1}{\log n \log m} \sum_{k=1}^{n-1} \sum_{l=1}^{m-1} \frac{S_{k,l}f(x,y)}{(n-k)(m-l)}.$$

The L^1 norm convergence holds in both cases for functions in $L(\log^+ L)^2$. Besides, this can not be improved in neither cases. That is, the situation in the second case is the same as in the "square case". The situation is not worse! This is highly surprising as we compare the Marcinkiewicz means and the two-dimensional Fejér means. The Fejér means:

$$\frac{1}{nm} \sum_{j=0}^{n-1} \sum_{k=0}^{m-1} S_{j,k}f.$$

The Marcinkiewicz means:

$$\frac{1}{n} \sum_{j=0}^{n-1} S_{j,j}f.$$

It is known that the Marcinkiewicz means of any integrable function converge to the function a.e. and in L^1 norm too [56]. In 2005 Gát and Goginava [22] proved (Walsh case), that if $\delta : [0,+\infty) \to [0,+\infty)$ measurable, $\delta(+\infty) = 0$, then there exists a two-variable $h \in L(\log^+ L)^2 \delta(L)(G^2)$ such that $t_{n,m}h$ does not converge h in $L(G^2)$ norm. Gát and Goginava proved (Walsh case) in another joint paper [21], the same result for the Nörlund logarithmic means of the square partial sums of the two-dimensional Fourier series. The proof of this theorem is more difficult then the proof for the two-dimensional Nörlund logarithmic means.

5.6 Representative product systems

Until the end of this article we deal with a generalization of Vilenkin groups an systems. The main idea is to take the complete direct product of arbitrary finite groups, even non-abelian groups. The characters of a finite non-abelian group can not form a complete system because they are less than the order of the group. The missing functions can be obtained computing the representations of the group. For this reason we introduce first some concepts of Representation Theory. The notation which we used is similar to the one appeared in [32].

Let $\mathcal{G}$ a not-necessarily abelian group. A *representation* U of a group $\mathcal{G}$ is a homomorphism of $\mathcal{G}$ into the semigroup of all operators defined in some linear space E over an arbitrary field F. That is, $U : x \to U_x$ such that $U_x : E \to E$ is a linear transformation for all $x \in \mathcal{G}$ and

$$U_{xy} = U_x U_y \qquad (x, y \in \mathcal{G}).$$

The linear space E is called the *representation* space of U, and let the dimension of a representation be the *dimension* of its own representation space. We can assume that U_e is the identity operator on E, because E is the direct sum of invariant subspaces E_0 and E_1 such that $U_x(E_0) = \{0\}$ for all $x \in G$, and U_e is the identity operator on E_1, hence we can take E by E_1.

Throughout this work suppose that the representation space of all representations is a reflexive complex Banach space which is a topological linear space under the metric and norm induced by the inner product $\langle ., . \rangle$. The representation U is called *unitary* if all of operators U_x are unitary, i.e. U_x is a linear isometry of E onto E. A representation U with representation space E is called *irreducible* if only the spaces $\{0\}$ and E are invariant under all operators U_x $(x \in \mathcal{G})$.

We can define an equivalence relation in the set of all continuous irreducible unitary representations of the group $\mathcal{G}$ in the following manner. Two representations U and U' with representation spaces E and E' respectively are equivalent if there is a bounded linear isometry $T : E \to E'$ such that

$$U'_x T = T U_x \qquad (x \in \mathcal{G}).$$

Denote by Σ the set of all equivalence classes induced by the above relation. Σ is called the *dual object* (Σ) of the group $\mathcal{G}$. The common dimension of all representations in the class $\sigma \in \Sigma$ is denoted by d_σ. All group have a trivial representation with dimension 1, namely the one which is identically equal to 1. A representation with dimension 1 is called a *character*, i.e. a character is a mapping $\chi : \mathcal{G} \to \mathbb{C}$ such that

$$\chi(xy) = \chi(x)\chi(y) \quad (x, y \in \mathcal{G}), \qquad |\chi(x)| = 1 \quad (x \in \mathcal{G}).$$

Theorem 5.6.1 *Let $\mathcal{G}$ be a finite group. Then*

(a) $|\Sigma|$ is equal to the number of conjugacy class in $\mathcal{G}$. (The system of the conjugacy classes is a partition of $\mathcal{G}$ induced by the equivalence relation: $a \sim b$ if and only if $\exists x \in \mathcal{G} : \ a = xbx^{-1}$).

(b) if $\Sigma = \{\sigma_1, \sigma_1 \dots \sigma_{|\Sigma|}\}$, then $|\mathcal{G}| = d^2_{\sigma_1} + d^2_{\sigma_2} + \cdots + d^2_{\sigma_{|\Sigma|}}$.

(c) d_{σ_i} is a divisor of $|\mathcal{G}|$ $(1 \le i \le |\Sigma|)$.

(d) if the group $\mathcal{G}$ is abelian, then $|\Sigma| = |\mathcal{G}|$ and all representations of $\mathcal{G}$ are characters.

(e) if the group $\mathcal{G}$ is not abelian, then there is a representation with dimension greater than 1.

The above properties of finite groups suggests to us the construction of d^2_σ numbers of functions by every $\sigma \in \Sigma$. Coordinate functions have just this property and are defined as follows. Let $U^{(\sigma)}$ be a continuous irreducible representation in the class σ of the dual object of $\mathcal{G}$. Functions

$$u^{(\sigma)}_{i,j}(x) := \langle U^{(\sigma)}_x \xi_i, \xi_j \rangle, \qquad i, j \in \{1, \dots, d_\sigma\}$$

are called *coordinate functions* for $U^{(\sigma)}$, where $\xi_1, \dots, \xi_{d_\sigma}$ is a fixed orthonormal basis in the representation space of $U^{(\sigma)}$.

Finite groups are compact groups with the discrete topology. Thus the normalized Haar measure μ on the compact finite groups $\mathcal{G}$ of order m has the following property: for every set $A \in \mathcal{G}$ and function $f : \mathcal{G} \to \mathbb{C}$

$$\mu_k(A) = \frac{|A|}{m} \qquad \text{and} \qquad \int_{\mathcal{G}} f \, d\mu = \frac{1}{m} \sum_{x \in \mathcal{G}} f(x).$$

One of the main theorem in harmonic analysis is the Weyl-Peter's Theorem.

Theorem 5.6.2 (Weyl-Peter) *Let $\mathcal{G}$ be a compact group with Haar measure μ. Then for all $\sigma \in \Sigma$ and $j,k \in \{1,2,\ldots,d_\sigma\}$ the set of functions $\sqrt{d_\sigma}u_{j,k}^{(\sigma)}$ is an orthonormal basis for $L^2(\mathcal{G})$. Thus for $f \in L^2(\mathcal{G})$, we have*

$$f = \sum_{\sigma\in\Sigma}\sum_{j,k=1}^{d_\sigma} d_\sigma \hat{f}(i,j,\sigma)u_{j,k}^{(\sigma)}, \tag{5.5}$$

where

$$\hat{f}(i,j,\sigma) := \int_{\mathcal{G}} f\overline{u_{j,k}^{(\sigma)}}\,d\mu$$

and the series in (5.5) converges in the metric of $L^2(\mathcal{G})$. Furthermore, if $\{a_{j,k}^{(\sigma)} : j,k \in \{1,2,\ldots,d_\sigma;\sigma\in\Sigma\}$ is any set of complex numbers such that

$$\sum_{\sigma\in\Sigma}\sum_{j,k=1}^{d_\sigma} d_\sigma |a_{j,k}^{(\sigma)}|^2 < \infty,$$

there is a unique function g in $L^2(\mathcal{G})$ such that $\hat{f}(i,j,\sigma) = a_{j,k}^{(\sigma)}$ for all $j,k \in \{1,2,\ldots,d_\sigma\}$; $\sigma \in \Sigma\}$ and for which accordingly

$$g = \sum_{\sigma\in\Sigma}\sum_{j,k=1}^{d_\sigma} d_\sigma a_{j,k}^{(\sigma)}u_{j,k}^{(\sigma)}.$$

The Weyl-Peter's Theorem assures us that if we normalize all coordinate functions of the finite group $\mathcal{G}$ in $L^2(\mathcal{G})$, we obtain an orthonormal and complete system. A normalized coordinate function is the product of the function and the square root of its respective dimension. Notice that if $\mathcal{G}$ is a commutative group then all of its representations are characters. Otherwise, the $\mathcal{G}$ has at least one representation with dimension greater than 1. Hence the L^∞-norm of normalized coordinate functions of these representations is greater than 1.

We take now a sequence of finite groups G_k with order m_k and dual object Σ_k, $(k \in \mathbb{N})$. Suppose that each group has discrete topology and normalized Haar measure μ_k. For a fixed $k \in \mathbb{N}$ let $\{\varphi_k^s : 0 \le s < m_k\}$ be a system of all normalized coordinate functions of G_k. We do not decide now the enumeration of the system φ, only suppose that φ_k^0 is always the character 1. Thus for every $0 \le s < m_k$ there exists a $\sigma \in \Sigma_k$, $i, j \in \{1, ..., d_\sigma\}$ such that

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \qquad (x \in G_k).$$

If the finite group G_k is commutative, then $d_\sigma = 1$ for all $\sigma \in \Sigma_k$ and the coordinate functions are characters.

Let G be the compact group formed by the complete direct product of G_k with the product of the topologies, operations and measures (μ), as well as the Vilenkin groups. So here we can define concepts and notations similar to those defined for Vilenkin groups, namely the concept of a bounded group, the expansion of a member of G, the M sequence and the expansion of a non-negative integer n. In order to simplicity we always use the multiplication to denote the group operation and use the symbol e to denote the identity of the groups.

Theorem 5.6.3 *Let G be the complete direct product of the finite groups G_k $(k \in \mathbb{N})$. Then*

(a) since G is compact, the set Σ is countable and the dimensions of all representations of G are finite.

(b) U is a continuous irreducible representation of G if and only if U is the tensor product of finite numbers of continuous irreducible representations of distinct groups G_k.

The above theorem helps us to construct an orthonormal and complete system on G. Indeed, by the tensor product, we only need to enumerate the finite product of functions appeared in different systems φ_k, where $k \in \mathbb{N}$. So we construct an orthonormal system like Vilenkin systems:

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x \in G),$$

where n is of the form $n = \sum_{k=0}^{\infty} n_k M_k$ and $x = (x_0, x_1, ...)$. Thus we say that ψ is the *representative product system* of φ. The Weyl-Peter's theorem assures us that the system ψ is orthonormal and complete on $L^2(G)$.

5.7 Convergence in L^p-norm of Fourier series

For representative product system we define the Fourier coefficients and partial sums of Fourier series just like (5.3) for Vilenkin systems. A basic problem of Fourier analysis is to obtain the values of p $(1 \le p < \infty)$ such that for all function $f \in L^p(G)$ the sequence of partial sums $S_n f$ of the Fourier series of f converges to the function f in L^p-norm. Convergence for $p = 2$ is obvious on an arbitrary group G since $L^2(G)$ is a Hilbert space. On the other hand, we obtain

Theorem 5.7.1 (Toledo) *For all G groups there exists a function $f \in L^1(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^1-norm.*

For Vilenkin-Fourier series we have convergence at the interval $1 < p < \infty$. However, we can not state the same for an arbitrary representative product system. In this regard, we study the simplest non-abelian structure, i.e. the complete product of the symmetric group on 3 elements.

Let $m_k = 6$ for all $k \in \mathbb{N}$ and $\mathbb{S}_3$ be the *symmetric group* on 3 elements. Let $G_k := \mathbb{S}_3$ for all $k \in \mathbb{N}$. $\mathbb{S}_3$ has two characters and a 2-dimensional representation. Using a calculation of the matrices corresponding to the 2-dimensional representation we construct the functions φ_k^s. In the notation the index k is omitted because all of the groups G_k are the same.

	e	(12)	(13)	(23)	(123)	(132)	$\lVert\varphi^s\rVert_1$	$\lVert\varphi^s\rVert_\infty$
φ^0	1	1	1	1	1	1	1	1
φ^1	1	-1	-1	-1	1	1	1	1
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^3	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$
φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$

Notice that the functions φ_k^s can take the value 0, and the product system of φ is not uniformly bounded. This facts encumber the study of this systems.

On the other hand,

$$\max_{0\le s<6} \|\varphi^s\|_1 \|\varphi^s\|_\infty = \frac{4}{3}.$$

For an arbitrary group G define

$$\Psi_k := \max_{n<M_k} \|\psi_n\|_1 \|\psi_n\|_\infty \qquad (k \in \mathbb{N}).$$

Ψ_k is the multiplication of the greatest product of the square root of the dimension and L^1-norm of the functions φ appeared in all group G_{m_j} for $0 \le j < k$. If G is the complete product of $\mathcal{S}_3$ we have

$$\Psi_k = \left(\frac{4}{3}\right)^k \to \infty$$

if $k \to \infty$. The boundedness of the sequence Ψ plays an important role in the norm convergence of Fourier series.

Theorem 5.7.2 (Toledo [51]) *If G is a bounded group with unbounded sequence Ψ and suppose that all the same finite groups appearing in the product of G have the same system φ at all of their occurrences. Then for all $p \neq 2$, $1 < p < \infty$ there exists a function $f \in L^p(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^p-norm.*

The complete product of $\mathcal{S}_3$ satisfies the conditions of the above theorem, so in this case we only can state for $p = 2$ that the n-th partial sums of Fourier series $S_n f$ converge to f in L^p-norm for all $f \in L^p(G)$. For other bounded cases not covered by Theorem 5.7.2, the convergence of Fourier series has not been proved. More result concerning this topic are appeared in [51].

It easy to construct a bounded group G with bounded sequence Ψ. Let $m_k = 8$ for all $k \in \mathbb{N}$ and $\mathcal{Q}_2$ be the the *quaternion group* of order 8, i.e.

$$\mathcal{Q}_2 := \{[a, b] : a^4 = e,\ b^2 = a^2,\ bab^{-1} = a^3\}.$$

Let $G_k = \mathcal{Q}_2$ for all $k \in \mathbb{N}$. $\mathcal{Q}_2$ has four characters and a 2-dimensional representation $(8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2)$. Using a calculation of the matrices corresponding to the 2-dimensional representation we construct the functions φ_k^s. In the notation the index k is also omitted.

	e	a	a^2	a^3	b	ab	a^2b	a^3b	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	-1	-1	-1	-1	1	1
φ^2	1	-1	1	-1	1	-1	1	-1	1	1
φ^3	1	-1	1	-1	-1	1	-1	1	1	1
φ^4	$\sqrt{2}$	$\sqrt{2}\imath$	$-\sqrt{2}$	$-\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^5	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^6	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^7	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}\imath$	$\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$

$\varphi^4,\dots,\varphi^7$ correspond to the 2-dimensional representation. Notice that values of $|\varphi^s|$ are 0 or the square of the corresponding dimension only. Hence, the absolute value of the (not normalized) coordinate functions are 0 or 1 respectively. Representations of this form are called *monomial representations*. If all of the representations are monomial, then $\Psi_k = 1$ for $k \in \mathbb{N}$.

Finally, we shall remark that the converge almost everywhere of Fourier series is yet an open problem.

5.8 Relation with the interval $[0,1[$

In [50] the author establishes a natural relation between the Haar integration on the complete direct product of finite discrete topological groups and the Lebesgue integration on the interval $[0,1[$. With this intention, order the elements of all G_k $(k \in \mathbb{N})$ groups in some way such that the first is always their identity. In fact, the ordering is a bijection between G_k and $\{0,1,\dots,m_k-1\}$ which give to every $x \in G_k$ the integer $0 \le \overline{x} < m_k$ $(\overline{e}=0)$. Define

$$|x| := \sum_{k=0}^{\infty} \frac{\overline{x_k}}{M_{k+1}} \qquad (x \in G).$$

It is easy to see that $|.|$ is a norm and the proceeded metric $d(x,y) := |xy^{-1}|$ induces the topology of G. In addition, $0 \leq |x| \leq 1$ for all $x \in G$. Using this fact we represent the group G in the interval $[0,1[$.

Any $x \in [0,1]$ can be written

$$x := \sum_{k=0}^{\infty} \frac{\overline{x_k}}{M_{k+1}} \qquad (0 \leq \overline{x_k} \leq m_k - 1),$$

but there are numbers with two expressions of this form. They are all numbers in the set

$$\mathbb{Q} := \left\{ \frac{p}{M_n} : 0 \leq p < M_n,\ n,p \in \mathbb{N} \right\}$$

called *m-adic rational numbers*. The other numbers have only one expression. The m-adic rational numbers have an expression terminates in 0's and other terminates in $m_k - 1$'s. We choose the first one to make an unique relation for all numbers in the interval $[0,1]$ with their expression, named de *m-adic expansion* of the number. In this manner we assign to a number in the interval $[0,1]$ having an m-adic expansion $(\overline{x_0}, \overline{x_1}, \dots)$ an element of G with expansion $(x_0, x_1, \dots)$ denoting this relation by ρ. ρ is called the *Fine's map*. Using Fine's map we introduce a new operation on the interval $[0,1[$:

$$x \odot y := |\rho(x)\rho(y)| \qquad (x,y \in [0,1[).$$

Let $L^0(G)$ denote the set of all measurable functions on G which are a.e. finite. In some way denote by L^0 the set of all Lebesgue measurable functions on $[0,1]$ which are a.e. finite. The following theorem shows the relation between the Haar integration on G and the Lebesgue integration on the interval $[0,1[$.

Theorem 5.8.1 (Toledo [50]) *Let ρ denote the Fine's map.*

(a) If $f \in L^0(G)$ then $f \circ \rho \in L^0$. Conversely, if $g \in L^0$ and

$$f(x) := g(|x|) \qquad (x \in G) \tag{5.6}$$

then $f \in L^0(G)$.

(b) If f is integrable on G then $f \circ \rho$ is Lebesgue integrable and

$$\int_G f\, d\mu = \int_0^1 (f \circ \rho)(x)\, dx.$$

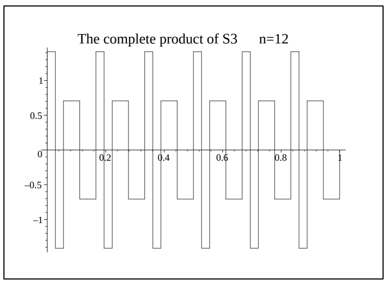
Conversely, if g is Lebesgue integrable and f is defined by (5.6) then f is integrable on G and

$$\int_0^1 g(x)\, dx = \int_G f\, d\mu.$$

According to Theorem 5.8.1, we can represent the system ψ on the interval $[0,1[$ substituting it by the

$$\upsilon_n := \psi_n \circ \rho \qquad (n \in \mathbb{N})$$

system. In Figure 1 we plot the corresponding values of ψ_{12} and ψ_{23} with respect to the complete product of $\mathcal{S}_3$. These graphs show two properties of the system ψ which are different to the commutative cases and difficult the study of the noncommutative cases: the system ψ is not uniformly bounded and can take the value 0.



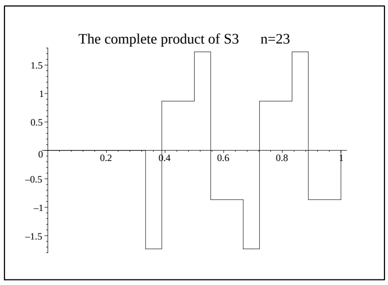


Figure 5.1: ψ_{12} and ψ_{23} with respect to the complete product of $\mathcal{S}_3$.

5.9 Dirichlet kernels

The *Dirichlet kernels* are defined as follows:

$$D_n(x,y) := \sum_{k=0}^{n-1} \psi_k(x)\overline{\psi}_k(y) \qquad (n \in \mathbb{N}).$$

It is easy to see that

$$S_n f(x) = \int_G f(y) D_n(x,y) d\mu(y), \tag{5.7}$$

which shows the importance of the Dirichlet kernels in the study of the convergence of Fourier series.

The set of intervals are given by $I_0(x) := G$,

$$I_n(x) := \{y \in G : y_k = x_k, \text{ for } 0 \le k < n\} \quad (x \in G, n \in \mathbb{P}).$$

The set of intervals $I_n := I_n(e)$ is a countable neighborhood base at the identity e of the product topology on G.

The authors proved in [26] the fact, that the Paley's lemma is also holds for non-abelian cases.

Lemma 5.9.1 (Paley's lemma) *If $n \in \mathbb{N}$ and $x, y \in G$, then*

$$D_{M_k}(x,y) = \begin{cases} M_k & \text{for } x \in I_k(y), \\ 0 & \text{for } x \notin I_k(y) \end{cases}$$

By Paley's lemma the operator $S_{M_n} f$ is the conditional expectation with respect to the σ-algebra generated by the sets $I_n(x)$, $x \in G$. So we obtain:

Theorem 5.9.1 *For all $1 \le p < \infty$ and $f \in L^p(G)$ the the partial sequence $S_{M_n} f$ with respect to any representative product system converge to f in L^p-norm and a.e.*

The above statement has even more significance in non-commutative cases. For functions belongs to $L^p(G)$ it is possible that the Fourier series do not converge to the function, but it has a partial sequence which already converges to the function, even for $p = 1$ and almost everywhere. This happens when G is the complete product of $\mathcal{S}_3$.

Paley's lemma holds for all representative product systems, but Dirichlet kernels are very different in non-abelian cases. We see the difference for instance if we consider the maximal value of Dirichlet kernels

$$\mathbb{D}_n := \sup_{x,\, y \in G} |D_n(x, y)| \qquad (n \in \mathbb{P}).$$

For Vilenkin systems $\mathbb{D}_n = n$ for all $n \in \mathbb{P}$, but the general case is a bit more different.

Theorem 5.9.2 (Toledo [52]) *If $n \in \mathbb{P}$ and $A := \max\{k \in \mathbb{N} : n_k \neq 0\}$, then*

$$n \leq \mathbb{D}_n \leq M_{A+1}.$$

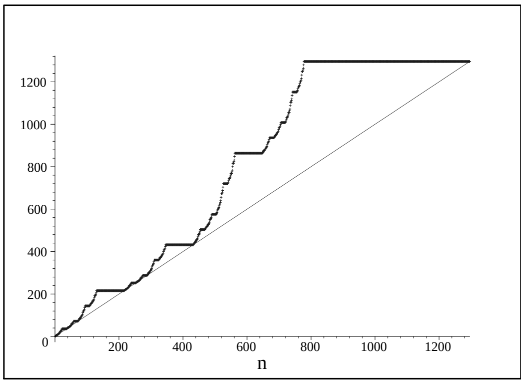


Figure 5.2: $\mathbb{D}_n$ $(n \leq 6^4)$ on the complete product of $\mathcal{S}_3$

5.10 Fejér means and Cesàro means of order α

We stated that For functions belongs to $L^p(G)$ $(1 \leq p < \infty)$ it is possible that the Fourier series do not converge to the function, even G is a bounded group. For this reason it is very interesting the fact that it is not possible for Fejér means, if G is a bounded group. Fejér means of Fourier series of the function f are given by

$$\sigma_n f = \frac{1}{n}\sum_{k=0}^{n-1} S_k f \qquad (n \in \mathbb{P}).$$

Thus, we have

Theorem 5.10.1 (Gát and Toledo [26]) *If G is a bounded group and $f \in L^p(G)$, $1 \leq p < \infty$, then $\sigma_n f \to f$ in L^p-norm.*

We state similar statements for the Cesàro means of order α. First we introduce the *Cesàro numbers of order α* given by the formula

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} \qquad (n \in \mathbb{N})$$

where α is a real number. We summarize the main properties of this numbers as follows (see [64]).

$$A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1}, \tag{5.8}$$

$$A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}, \tag{5.9}$$

$$\lim_{n\to\infty} \frac{A_n^\alpha}{n^\alpha} = \frac{1}{\Gamma(\alpha+1)} \qquad (\alpha \neq -1, -2, \dots), \tag{5.10}$$

the numbers A_n^α are positive if $\alpha > -1$, and $A_n^\alpha < 1$ if $-1 < \alpha < 0$, (5.11)

the sequence A_n^α increasing for $\alpha > 0$ and decreasing for $-1 < \alpha < 0$. (5.12)

Using the notation above we denote *the Cesàro means of order* α *of Fourier series* or simply (C,α) *means* by

$$\sigma_{n+1}^{\alpha} f := \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f \qquad (n \in \mathbb{P}). \tag{5.13}$$

It is not difficult to see that

$$\sigma_n^0 f = S_n f \qquad \text{and} \qquad \sigma_n^1 f = \sigma_n f.$$

Theorem 5.10.2 (Gát and Toledo [27]) *Let* G *be a bounded group,*

$$\alpha_0 := \limsup_{k\to\infty} \log_{m_k} \left(\max_{0\le s<m_k} \|\varphi_k^s\|_1 \|\varphi_k^s\|_\infty \right),$$

$\alpha_0 < \alpha < 1$ *and* $f \in L^p(G)$ *for* $1 \le p < \infty$*. Then* $\sigma_n^\alpha f \to f$ *in* L^p*-norm.*

We remark that the number α_0 exists and it less than $\frac{1}{2}$ since $\|\varphi_k^s\|_\infty^2 < m_k$ and $\|\varphi_k^s\|_1 \le 1$ for all $k \in \mathbb{N}$. On the other hand, if the group G are monomial the property $\|\varphi_k^s\|_1 \|\varphi_k^s\|_\infty = 1$ implies $\alpha_0 = 0$. Hence we obtain immediately the next corollary for monomials and so for commutative cases.

Corollary 5.10.1 *If* G *is a bounded monomial group,* $0 < \alpha < 1$ *and* $f \in L^p(G)$*,* $1 \le p < \infty$*, then* $\sigma_n^\alpha f \to f$ *in* L^p*-norm.*

It is not difficult to calculate that $\alpha_0 = \log_6 \frac{4}{3}$ for the complete product of $\mathcal{S}_3$. Thus we obtain:

Corollary 5.10.2 *Let* G *be the complete product of* $\mathcal{S}_3$*. If* $f \in L^p(G)$*,* $1 \le p < \infty$ *and* $\alpha > \log_6 \frac{4}{3}$*, then* $\sigma_n^\alpha f$ *converge to the function* f *in* L^p*-norm.*

On the other hand, suppose we have a bounded group G with $\alpha_0 > 0$. Thus we obtain divergence in the following case.

Theorem 5.10.3 (Gát and Toledo [27]) *Let G be a bounded group,*

$$\alpha_1 := \liminf_{k\to\infty} \log_{m_k}\left(\max_{0\le s<m_k} \|\varphi_k^s\|_1\|\varphi_k^s\|_\infty\right),$$

and $0 < \alpha < \alpha_1$. Then there exists an $f \in L^1(G)$ such that $\sigma_n^\alpha f$ does not converge to the function f in L^1-norm.

Finally, we remark that if G is the complete product of the same finite group with the same system φ, then $\alpha_1 = \alpha_0$. Hence we obtain:

Corollary 5.10.3 (see [27]) *Let G be the complete product of $\mathcal{S}_3$. If $\alpha < \log_6 \frac{4}{3}$, then there exists an $f \in L^1(G)$ such that $\sigma_n^\alpha f$ does not converge to the function f in L^1-norm.*

5.11 References

[1] G. H. Agaev, N. Ja. Vilenkin, G. M. Dzsafarli, and A. I. Rubinstein, *Multiplicative systems of functions and harmonic analysis on 0-dimensional groups*, Izd. (ELM), Baku, 1981.

[2] E.S. Belinsky, On the summability of Fourier series with the method of lacunary arithmetic means, *A*nal. Math. , 10:275-282, 1984.

[3] E.S. Belinsky, Summability of Fourier series with the method of lacunary arithmetical means at the Lebesgue points, *Proc. Am. Math. Soc.*, 125(12):3689-3693, 1997.

[4] G. Benke, Trigonometric approximation theory in compact totally disconnected groups, *Pacific J. of Math.*, 77(1):23-32, 1978.

[5] P. Billard, Sur la convergence presque partout des séries de Fourier-Walsh des fonctions de l'espace $L^2(0,1)$, *Studia Math.*, 28:363388, 1967.

[6] I. Blahota and G. Gát, Norm summability of Nörlund logarithmic means on unbounded Vilenkin groups, *Anal. Theory Appl.*, 24(1):1-17, 2008.

[7] L. Carleson, Appendix to the paper by J.-P. Kahane and Y. Katznelson, *Series de Fourier des fonctions bornees*, Studies in pure mathematics, Birkhauser, Basel-Boston, Mass., pages 395-413, 1983.

[8] N.J. Fine, Cesàro summability of Walsh-Fourier series, *Proc. Nat. Acad. Sci. U.S.A.*, 41:558-591, 1955.

[9] S. Fridli, On the modulus of continuity with respect to functions defined on Vilenkin groups, *Acta Mat. Hungar.*, 45(3-4):393-396, 1985.

[10] G. Gát, Pointwise convergence of the Cesàro means of double Walsh series, *Ann. Univ. Sci. Budap. Rolando Eoetvoes, Sect. Comput.*, 16:173-184, 1996.

[11] G. Gát, Pointwise convergence of the Fejér means of functions on unbounded Vilenkin groups, *J. Approx. Theory*, 101(1):1-36, 1999.

[12] G. Gát. On the divergence of the $(C, 1)$ means of double Walsh-Fourier series, *Proc. Am. Math. Soc.*, 128(6):1711-1720, 2000.

[13] G. Gát, Divergence of the $(C, 1)$ means of d-dimensional Walsh-Fourier series, *Anal. Math.*, 27(3):157-171, 2001.

[14] G. Gát. Cesàro means of integrable functions with respect to unbounded Vilenkin systems, *J. Approx. Theory*, 124(1):25-43, 2003

[15] G. Gát. Convergence of Marcinkiewicz means of integrable functions with respect to two-dimensional Vilenkin systems, *Georgian Math. Journal*, 10(3) : 467-478, 2004.

[16] G. Gát. On the pointwise convergence of Cesŕro means of two-variable functions with respect to unbounded Vilenkin systems, *J. Approx. Theory*, 128(1):69-99, 2004.

[17] G. Gát, On the divergence of the Fejér means of integrable functions on twodimensional Vilenkin groups, *Acta Math. Hung.*, 107(1-2):17-33, 2005.

[18] G. Gát. Almost everywhere convergence of Fejér means of L^1 functions on rarely unbounded Vilenkin groups, *Acta Math. Sin.* , Engl. Ser., 23 (12) : 2269-2294, 2007.

[19] G. Gát, Almost everywhere convergence of fejér and logarithmic means of subsequences of partial sums of the Walsh-Fourier series of integrable functions, *J. Approx. Theory*, 162(4):687-708, 2010.

[20] G. Gát and U. Godinava, Uniform and L-convergence of logarithmic means of Walsh-Fourier series, *Acta Math. Sin.*, Engl. Ser., 22(2):497-506, 2006.

[21] G. Gát and U. Goginava, Uniform and L-convergence of logarithmic means of cubical partial sums of doubleWalsh-Fourier series, *East J. Approx.*, 10(4) : 391-412, 2004.

[22] G. Gát and U. Goginava, Uniform and L-convergence of logarithmic means of double Walsh-Fourier series, *Georgian Math. J.*, 12(1):75-88, 2005.

[23] G. Gát and U. Goginava, Maximal convergence space of a subsequence of the logarithmic means of rectangular partial sums of double Walsh-Fourier series, *Real Anal. Exch.*, 31(2):447-464, 2006.

[24] G. Gát and U. Goginava, Uniform and L-convergence of logarithmic means of Walsh-Fourier series, *Acta Math. Sin., Engl. Ser.*, 22(2):497-506, 2006.

[25] G. Gát, U. Goginava, and G. Tkebuchava, Convergence in measure of logarithmic means of quadratical partial sums of double Walsh-Fourier series, *J. Math. Anal. Appl.*, 323(1):535-549, 2006.

[26] G. Gát and R. Toledo. Lp-norm convergence of series in compact totally disconected groups, *Anal. Math.*, 22:13-24, 1996.

[27] G. Gát and R. Toledo, On the converge in L1-norm of Cesàro means with respect to representative product systems, *Acta Math. Hungar.*, 123(1-2):103-120, 2009.

[28] V.A. Glukhov, Summation of Fourier-Walsh series, *Ukr. Math. J.*, 38:261-266, 1986.

[29] V.A. Glukhov, Summation of multiple Fourier series in multiplicative systems, *Math. Notes*, 39:364-369, 1986.

[30] U. Goginava, Almost everywhere summability of multiple Fourier series, *Math. Anal. and Appl.*, 287(1):90-100, 2003.

[31] U. Goginava, Marcinkiewicz-Fejér means of d-dimensional Walsh-Fourier series, *J. Math. Anal. Appl.*, 307(1):206-218, 2005.

[32] E. Hewitt and K. Ross, Abstract Harmonic Analysis I. Springer-Verlag, Heidelberg, 1963.

[33] J. A. Zygmund J. Marcinkiewicz, On the summability of double Fourier series, *Fund. Math.*, 32:122-132, 1939.

[34] S.V. Konyagin, The Fourier-Walsh subsequence of partial sums, *Math. Notes*, 54(4):1026-1030, 1993.

[35] N.R. Ladhawala and D.C. Pankratz, Almost everywhere convergence of Walsh Fourier series of H_1-functions, *Stud. Math.*, 59:37-92, 1976.

[36] J. Marcinkiewicz, Quelques théorčmes sur les séries orthogonales, *Ann Soc. Polon. Math.*, 16:85-96, 1937.

[37] F. Móricz and A.H. Siddiqi, Approximation by Nörlund means of Walsh- Fourier series, *J. Approx. Theory*, 70(3):375-389, 1992.

[38] J. Neveu, *Martingales à temps discret*, Masson et Cie, Paris, 1972.

[39] R.$\check{Z}$. Nurpeisov, On the uniform $(C, 1)$ summability of Fourier series and integrability of series with respect to multiplicative orthonormal systems, *Analysis Math.*, 15:127-143, 1989.

[40] J. Pál and P. Simon, On a generalization of the concept of derivative, *Acta Math. Acad. Sci. Hungar.*, 29:155-164, 1977.

[41] L. S. , Topological Groups. Gordon and Breach, Science Publishers, Inc., New York, 1966.

[42] J. Price, Certain groups of orthonormal step functions, *Canadian J. Math.*, 9:413-425, 1957.

[43] A.I. Rubinshtein, On the modulus of continuity of functions defined on 0- dimensional groups, *Mat. Zametki*, 23:379-388, 1978.

[44] R. Salem, On strong summability of Fourier series, *Am. J. Math.*, 77:393-403, 1955.

[45] F. Schipp, Über gewiessen Maximaloperatoren, *Annales Univ. Sci. Budapestiensis, Sectio Math.*, 18:189-195, 1975.

[46] F. Schipp, On L^p-norm convergence of series with respect to product systems, *Analysis Math.*, 2:49-63, 1976.

[47] P. Simon, Cesàro means of Vilenkin-Fourier series, *Publ. Math.*, 59(1-2):203-219, 1976.

[48] P. Simon, Verallgemeinerte Walsh-Fourierreihen II, *Acta. Math. Acad. Sci. Hungar.* , 27:329-341, 1976.

[49] M.H. Taibleson, Fourier Series on the Ring of Integers in a p-series Field, *Bull. Amer. Math. Soc.*, 73:623-629, 1967.

[50] R. Toledo, Representation of product systems on the interval $[0, 1]$, *Acta Math. Acad. Paed. Nyíregyháza*, (19/1):43-50, 2003.

[51] R. Toledo, Negative results concerning Fourier series on the complete product of S_3, *J. Inequal. Pure and Appl. Math.*, 9(4), Art. 99:7 pp, 2008.

[52] R. Toledo, On the maximal value of Dirichlet kernels with respect to representative product systems, In Proceedings of 6th International Conference on Functional Analysis and Approximation Theory, Rendiconti del Circolo Matematico di Palermo, volume 82 of Serie II, Suppl., pages 431-447, 2010.

[53] N. Ja. Vilenkin, On a class of complete orthonormal system, *Izv. Akad. Nauk SSSR, ser. mat.*, 11:363-400, 1947.

[54] W.R. Wade, Growth of Ces*acutea*ro means of double Vilenkin-Fourier series of unbounded type. Analysis of divergence, Control and management of divergent processes. Proceedings of the 7th international workshop in analysis and its applications, IWAA, Orono, ME, USA June 1-6, 1997. Boston, MA: Birkhauser. Applied and Numerical Harmonic Analysis, pages 41-50, 1999.

[55] F. Weisz, Cesáro summability of two-dimensional Walsh-Fourier series, *Trans. Amer. Math. Soc.*, 348:2169-2181, 1996.

[56] F. Weisz, A generalization for Fourier transforms of a theorem due to Marcinkiewicz, *J. Math. Anal. Appl.*, 252(2):675-695, 2000.

[57] F. Weisz, Summability results of Walsh- and Vilenkin-Fourier series. Leindler, L. (ed.) et al., Functions, series, operators. Alexits memorial conference in honor of the 100th anniversary of the birth of Professor George Alexits (1899-1978), Budapest, Hungary, August 9-13, 1999. Budapest: János Bolyai Mathematical Society. 443-464 (2002)., 2002.

[58] F. Weisz, Marcinkiewicz-θ-summability of double Fourier series, *Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Comput.*, 24:103-118, 2004.

[59] F. Weisz, Strong Marcinkiewicz summability of multi-dimensional Fourier series, *Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Comput.*, 29:297-317, 2008.

[60] W.S. Young. Mean convergence of generalized Walsh-Fourier series, *Trans. Amer. Math. Soc.*, 218:311-320, 1976.

[61] N.A. Zagorodnij and R.M. Trigub, A question of Salem, Theory of functions and mappings. Collect. sci. Works, Kiev, 1979.

[62] Z. Zalcwasser, Sur la sommabilité des séries de Fourier, *Stud. Math.*, 6:82-88, 1936.

[63] L.V. Zhizhiasvili, Generalization of a theorem of Marcinkiewicz, *Izv. Akad. nauk USSR Ser Mat.*, 32:1112-1122, 1968.

[64] A. Zygmund. Trigonometric Series. University Press, Cambridge, 1959.

[65] A. Zygmund. Trigonometric series. Vol. I and II. 2nd reprint of the 2nd ed. Cambridge etc., Cambridge University Press., 1977.

6

Fredholm theory and Kato decomposition

G. Kantún-Montiel[1], S.V. Djordjević [2] and R. Harte[3]

Abstract: We present several Kato type decompositions and study some relations with Fredholm theory.

Keywords: Fredholm operator, Riesz projection, Kato decomposition

AMS Subject Classification: 47A53, 47A25

6.1 Introduction

Let X be an arbitrary complex Banach space. By $B(X)$ we will denote the set of bounded linear operators from X into X. For an operator $T \in B(X)$ and a subspace $M \subseteq X$ we write

$$T(M) = \{y \in X : \exists x \in M, Tx = y\}.$$

[1]Universidad de Navojoa, gkantun@mexico.com

[2]Benemérita Universidad Autónoma de Puebla, slavdj@fcfm.buap.mx

[3]Trinity College Dublin, rharte@maths.tcd.ie

In particular, we write $R(T) = T(X)$ and we call this the range of T. If $R(T) = X$ we say that T is surjective. In a similar way, we write

$$T^{-1}(M) = \{x \in X : Tx \in M\}.$$

In particular, we write $N(T) = T^{-1}(0)$ and we call this the null space of T. If $N(T) = \{0\}$ we say that T is injective.

Let M and N be subspaces of X. If for every $x \in X$ there exists $u \in M$ and $v \in N$ such that $x = u + v$ we say that X is the *sum* of M and N, denoted $X = M + N$. If M and N are also closed and $M \cap N = \{0\}$ we say that X is the (topological) *direct sum* of M and N, denoted by $X = M \oplus N$. In this case, we say that M is (topologically) *complemented* and that N is the (topological) *complement* of M. Analogously N is complemented by M.

If there exist subspaces M, N of X such that $X = M \oplus N$ we say that the pair (M, N) *decomposes* X and that (M, N) is a *decomposition* of X. If M is finite dimensional (denoted $\dim M < \infty$) then it is closed and complemented. If $M \subset X$ is not finite dimensional, then we can not guarantee that M is complemented, even if it is closed.

In our study, we will find the "projection" operators very helpful. An operator $P \in B(X)$ is a *projection* if $P = P^2$. Let $x \in R(P)$, then there exists $y \in X$ such that $x = Py$. Hence, $Px = PPy = P^2y = Py = x$.

We see that if P is a projection, then $I - P$ also is, where I denotes the identity operator $Ix = x$. Also, if $x \in R(P)$, then $Px = x$ and $(I - P)x = x - Px = 0$, thus $R(P) = N(I - P)$. In the same way, $N(P) = R(I - P)$.

Theorem 6.1.1 ([18, Lemma 5.64]) *For the subspaces M and N of X, the following statements are equivalent:*
(1) $X = M \oplus N$.
(2) There exist a projection $P \in B(X)$ such that $R(P) = M$ and $N(P) = N$.

Proof. (1)$\Rightarrow$ (2). Let P be defined by $Px = x$ if $x \in M$ and $Px = 0$ if $x \in N$. It is clear that $R(P) = M$, $N(P) = N$ and since $P^2x = PPx = Px$, we have $P = P^2$. To see that $P \in B(X)$, by using the Closed Graph Theorem, we have to prove that the graph of P is closed.

Let $(x_n, Px_n)_n$ be a sequence in the graph $G(P)$ of P such that it converges to (x, y) in $X \times X$. To see that $G(P)$ is closed we have to prove that (x, y) is in $G(P)$, that is, $y = Px$. Since M is closed and Px_n converges to y, we have that $y \in M$. Moreover, $x_n - Px_n = (I - P)x_n$ is in N for every n, hence $x - y = \lim_{n\to\infty}(x_n - Px_n)_n$ is in N since N is closed. Therefore,

$$0 = P(x - y) = Px - Py = Px - y.$$

(2)⇒ (1). Let $x \in X$, and let $u = Px$ and $v = (I - P)x$. Then, $x = Px + x - Px = u + v$, $u \in R(P)$ and $v \in R(I - P) = N(P)$. Thus, $X = M + N$. Let $x \in R(P) \cap N(P)$, we have $x = Px = 0$, so $M \cap N = \{0\}$. Since P is continuous, $N(P) = N$ is closed. Also $I - P$ is continuous, so $N(I - P) = R(P) = M$ is closed.□

As we saw in the above theorem, the mapping $x \mapsto u = Px$ from X into X is continuous, hence the name of topological direct sum and topological complement. This should not be confused with the algebraic direct sum. Every subspace is algebraically complemented, but it is not, in general, in the topological sense we have been dealing with.

Let $T \in B(X)$. We say that a subspace $M \subset X$ is T invariant if $T(M) \subset M$. For a subset $M \subset X$, we denote by $T|_M$ the *restriction* of T to the subspace M given by $T|_M x = Tx$ for every $x \in M$.

Let (M, N) be a decomposition of X. If M and N are T invariant, then we say that (M, N) *decomposes* T. Let $T_1 = T|_M : M \to M$ and $T_2 = T|_N : N \to N$. We say that (T_1, T_2) is a *decompositon* of T, which is also denoted $T = T_1 \oplus T_2$.

Let $S, T \in B(X)$. Let us suppose that $ST = TS$. Let $x \in R(S)$, then, there exist $y \in X$ such that $x = Sy$. Thus, $Tx = TSy = STy$, and from this we see that $T(R(S)) \subset R(S)$. Now suppose that $T \in B(X)$ commutes with $P = P^2 \in B(X)$. Then T also commutes with $I - P$. If $M = R(P)$ and $N = N(P)$ we have

$$T(M) \subset M,$$

$$T(N) = T(R(I - P)) \subset R(I - P) = N.$$

Generally speaking, the present study is based on the idea of decomposing an operator T in two "simpler" operators in a way we get some advantages in practical computations and in theoretical understanding of such operator. In section 2 we will decompose an operator in two operators with prescribed eigenvalues, or more generally spectral sets. For instance, decomposing an operator in an invertible one and another which is not. In section 3 we will decompose in a quasinilpotent operator and a Fredholm or, more generally, semi-Regular one. In section 4 we will study the relationship between Fredholm theory and invertibility, which will enable us to see the decompositions in the light of generalized notions of invertibility in the Calkin algebra. In section 5 we will deal with the algebraic properties, without reference to the topology of X, used in above sections, thus extending the results to general Banach algebras.

6.2 The spectral decomposition

An operator $T \in B(X)$ is *invertible* if there is some operator $S \in B(X)$ such that $TS = ST = I$, where I is the identity operator. For an invertible operator $T \in B(X)$, we will denote its inverse by T^{-1} (not to be confused with the notation $T^{-1}(M)$ used in above section). For a bounded linear operator acting on a Banach space, via open mapping theorem, to be invertible is equivalent to be surjective and injective.

Let $T \in B(X)$. If there exist $\lambda \in \mathbb{C}$ such that

$$Tx = \lambda x$$

for some $x \in X$, $x \neq 0$, we say that λ is an eigenvalue for T, and that x is an eigenvector associated to the eigenvalue λ. Let us note that in this case

$$(T - \lambda I)x = 0,$$

that is, $x \in N(T-\lambda I)$, $x \neq 0$, and hence the operator $T-\lambda I$ is not invertible.

More generally, let us consider the set of points λ in the complex plane for which $T - \lambda I$ is not invertible. Such a set will be called the *spectrum* of T:

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

The set of points λ in the complex plane for which $T - \lambda I$ is invertible will be called the *resolvent set* of T:

$$\rho(T) = \mathbb{C} \setminus \sigma(T).$$

For $\lambda \in \rho(T)$ the *resolvent function* associated with T is

$$R(T, \lambda) = (T - \lambda I)^{-1}.$$

The spectrum of a bounded linear operator is a closed, bounded and non void set of the complex plane (unless that $X = \{0\}$). The resolvent function is holomorphic in the resolvent set of the operator.

A set $\Lambda \subset \sigma(T)$ is a *spectral set* for T if Λ is open and closed in the relative topology. Note that a point $\lambda \in \sigma(T)$ is a spectral set if and only if it is an isolated point of $\sigma(T)$.

Let C be a Jordan curve. The important thing about Jordan curves is that they have defined the notion of interior and exterior (in a way we won't discuss here). Let us denote by $\mathrm{int}(C)$ the interior of C and by $\mathrm{ext}(C)$ the exterior. If for some spectral set Λ we have that $\Lambda \subset \mathrm{int}(C)$ and $\sigma(T) \setminus \Lambda \subset \mathrm{ext}(C)$ holds we say that C *separates* Λ from $\sigma(T) \setminus \Lambda$.

However, the Jordan curves are not enough:

Example 6.2.1 ([1, Example 1.20]**)** There is an operator for which there is no Jordan curve separating a spectral set Λ from $\sigma(T) \setminus \Lambda$.

Let $X = \ell^2$ be the space of square summable sequences, and let $\{\theta_1, \theta_2, \theta_3, \ldots\} = [0, 1] \cap \mathbb{Q}$. Define an operator T by

$$Tx = (0, 2x_2, e^{2\pi i\theta_1}x_3, e^{2\pi i\theta_2}x_4, e^{2\pi i\theta_3}x_5, \ldots), \qquad x = (x_1, x_2, x_3, \ldots).$$

We have that 0, 2 and $e^{2\pi i\theta_k}$ are all the eigenvalues of T. Since the spectrum is a compact set, we see that

$$\sigma(T) = \{0\} \cup \{2\} \cup \{\lambda : |\lambda| = 1\}.$$

Take $\Lambda = \{\lambda : |\lambda| = 1\}$, then, there is no Jordan curve separating Λ from $\{0\} \cup \{1\}$.

To overcome this problem, we need the concept of "Cauchy contour".

An *elementary Cauchy domain* is a bounded, open and connected subset of $\mathbb{C}$ whose boundary is the union of a finite number of non intersecting Jordan curves. A finite union of elementary Cauchy domains with disjoint closures will be called a *Cauchy domain*.

Let D be a Cauchy domain. If each Jordan curve involved in the boundary of D is oriented in such a way that points in D lie to the left as the curve is traced out, then this oriented boundary is called a *Cauchy contour*.

Let C be a Cauchy contour corresponding to a Cauchy domain D. The interior of C, denoted $\text{int}(C)$, is the Cauchy domain D and the exterior of C, denoted $\text{ext}(C)$, is the set $\mathbb{C} \setminus (D \cup C)$.

Let $E, \tilde{E} \subset \mathbb{C}$, we say that C separates E from $\tilde{E}$ if $E \subset \text{int}(C)$ and $\tilde{E} \subset \text{ext}(C)$. The set of all Cauchy contours separating a spectral set Λ from the set $\sigma(T) \setminus \Lambda$ will be denoted by $C(T, \Lambda)$.

Lemma 6.2.1 ([1, Corollary 1.22]) *Let E be a compact subset of $\mathbb{C}$ and let $\widetilde{E}$ be a closed subset of $\mathbb{C}$. If $E \cap \widetilde{E} = \emptyset$ then there exist a Cauchy contour separating E from $\widetilde{E}$.*

For a spectral set Λ for T and $C \in C(T, \Lambda)$, we define

$$P = P(T, \Lambda) = -\frac{1}{2\pi i} \int_C R(T, z) dz.$$

This mapping $P(T, \Lambda)$ is known as the spectral projection of T corresponding to the spectral set Λ.

Theorem 6.2.1 ([1, Proposition 1.23]) *P is a bounded projection.*

Proof. Let $C \in \mathcal{C}(T, \Lambda)$. By Lemma 6.2.1 there is a Cauchy contour $\tilde{C}$ such that $\Lambda \cup C \subset \text{int}\tilde{C}$ and $\sigma(T) \setminus \Lambda \subset \text{ext}\tilde{C}$. Let us denote $R(z) = R(T, z)$ for $z \in \mathbb{C}$. We have

$$P^2 = \left(-\frac{1}{2\pi i}\right)^2 \int_C \left[\int_{\tilde{C}} R(z) R(\tilde{z}) d\tilde{z}\right] dz.$$

Using the first resolvent identity we get

$$R(z) - R(\tilde{z}) = (z - \tilde{z})R(z)R(\tilde{z}) \qquad \text{for } z \in C, \tilde{z} \in \tilde{C}.$$

Thus,

$$\begin{aligned}
P^2 &= \left(-\frac{1}{2\pi i}\right)^2 \int_C \left[\int_{\tilde{C}} [R(z) - R(\tilde{z})] \frac{d\tilde{z}}{z - \tilde{z}}\right] dz \\
&= \left(-\frac{1}{2\pi i}\right)^2 \int_C \left[R(z) \int_{\tilde{C}} \frac{d\tilde{z}}{z - \tilde{z}}\right] dz \\
&\quad - \left(-\frac{1}{2\pi i}\right)^2 \int_{\tilde{C}} \left[R(\tilde{z}) \int_C \frac{dz}{z - \tilde{z}}\right] d\tilde{z}
\end{aligned}$$

And since $z \in C \subset \text{int}\tilde{C}$ and $\tilde{z} \in \tilde{C}$ we have, using Cauchy's theorem,

$$\int_{\tilde{C}} \frac{d\tilde{z}}{z - \tilde{z}} = -2\pi i, \qquad \int_C \frac{dz}{z - \tilde{z}} = 0.$$

Hence, $P = P^2$. Further, $P \in B(X)$ since it is a contour integral of a $B(X)$-valued function.□

Having now a bounded projection, we know by the previous section how to make a decomposition of X. Moreover, since

$$\begin{aligned}
T(T - \lambda I)^{-1} &= (T - \lambda I)^{-1}(T - \lambda I)T(T - \lambda I)^{-1} \\
&= (T - \lambda I)^{-1}T(T - \lambda I)(T - \lambda I)^{-1} \\
&= (T - \lambda I)^{-1}T.
\end{aligned}$$

we have that $(R(P), N(P))$ decomposes T. What is special about this decomposition is that it is related with the spectral set corresponding to the spectral projection P as follows.

Theorem 6.2.2 ([1, Theorem 1.26]) *Let $P = P(T, \Lambda)$ with $T \in B(X)$ and let Λ be a spectral set for T. Let $M = R(P)$, $N = N(P)$, then*

$$T(M) \subset M, \qquad T(N) \subset N,$$

$$\sigma(T|_M) = \Lambda, \qquad \sigma(T|_N) = \sigma(T) \setminus \Lambda.$$

In particular, if 0 is an isolated point of the spectrum of T, then the decomposition $(R(P), N(P))$, where $P = P(T, \{0\})$, decomposes T in an invertible operator and an operator whose spectrum consists of the point 0 alone. The isolated spectral points are interesting because we can say something more about the range and kernel of the spectral projection corresponding to such isolated point.

Let λ be an isolated point of $\sigma(T)$. The *algebraic multiplicity* of λ is $\dim R(P(T, \{\lambda\}))$. The isolated eigenvalues of T with finite algebraic multiplicity are called *Riesz points*.

Theorem 6.2.3 ([1, Proposition 1.31]) *Let λ be an isolated point of $\sigma(T)$ and let P be the corresponding spectral projection. Let $M = R(P)$ and $N = N(P)$. If λ is a Riesz point of algebraic multiplicity m, then*

$$M = N((T - \lambda I)^m), \qquad N = R((T - \lambda I)^m),$$

and

$$X = N((T - \lambda I)^m) \oplus R((T - \lambda I)^m).$$

Proof. We will prove $N((T - \lambda I)^m) \subset M$. Let $x \in N((T - \lambda I)^m)$. Since T and P commute, we have

$$(T - \lambda I)^m (x - Px) = 0.$$

But $x - Px \in N$, and since $(T - \lambda I)^m|_N$ is injective (in fact invertible with inverse $-\frac{1}{2\pi i}\int_C (T - zI)^{-1} \frac{dz}{\lambda - z}$), we have that $x - Px = 0$, that is, $x \in M$.

Now we prove $M \subset N((T - \lambda I)^m)$. Since $\dim M = m$, the operator $T|_M : M \to M$ can be written, with respect to some ordered basis, as an upper triangular matrix $m \times m$ whose only eigenvalue is λ. Hence, $((T - \lambda I)_M)^m = O|_M$. Thus, $M \subset ((T - \lambda I)|_M)^m \subset ((T - \lambda I))^m$.

We will show $N \subset R((T - \lambda I)^m)$. Let $x \in N$. Since $(T - \lambda I)^m|_N$ is invertible, there exists $y \in N$ such that

$$x = (T - \lambda I)^m|_N y = (T - \lambda I)^m y.$$

Hence, $x \in R((T - \lambda I)^m)$.

Now we prove $R((T-\lambda I)^m) \subset N$. Let $x \in R((T-\lambda I)^m)$. Then there exists $y \in X$ such that $x = (T-\lambda I)^m y$. Since P commutes with T and $Py \in M$, we get

$$Px = P(T-\lambda I)^m y = (T-\lambda I)^m Py = O.$$

Therefore $x \in N(P) = N$.

Finally, since $X = M \oplus N$, we have $X = N((T-\lambda I)^m) \oplus R((T-\lambda I)^m)$.□

6.3 The Kato decomposition

We will begin by defining some operator classes which will be useful in later discussion.

An operator $T \in B(X)$ is *nilpotent* if there exists some $n \in \mathbb{N}$ such that $T^n = O$, where O is the operator identically zero $Ox = 0$. If T is nilpotent, then $\sigma(T) = \{0\}$. However, it may happen that the spectrum of an operator consists of the point zero alone but the operator is not nilpotent. Such an operator is called *quasinilpotent*.

We say that $T \in B(X)$ is *semi-regular* if $R(T)$ is closed and $N(T) \subset R(T^n)$ for every $n \in \mathbb{N}$. It is clear that invertible operators are semi-regular.

Now we discuss a little about the range and kernel of the powers of an operator. Let us define $T^0 = I$. It is not hard to see that

$$R(T^0) \supseteq R(T) \supseteq R(T^2) \supseteq \dots \quad \text{y} \quad N(T^0) \subseteq N(T) \subseteq N(T^2) \subseteq \dots$$

The *hyperrange* and the *hyperkernel* of an operator are defined in the following way:

$$R^\infty(T) = \bigcap R(T^n), \quad N^\infty(T) = \bigcup N(T^n), \quad n \in \mathbb{N}.$$

Thus, an operator is semi-regular if $N(T) \subset R^\infty(T)$.

It is interesting that, being T semi-regular or not, for $\lambda \neq 0$ we have

$$N(T+\lambda I) \subseteq N^\infty(T+\lambda I) \subseteq R^\infty(T).$$

In the previous section we saw how to decompose an operator into one invertible and another quasinilpotent. In the Kato type decompositions we are about to study in this section the idea is to decompose into quasinilpotent and another operator which is not necessarily invertible but which has some "good" properties.

We say that T admits a *generalized Kato decomposition* if there exist T-invariant closed subspaces $M, N \subset X$ such that

1. $X = M \oplus N$,
2. $T|_M$ is semi-regular and
3. $T|_N$ is quasinilpotent.

If $T|_N$ is nilpotent, we say that T admits a *Kato decomposition*, and if d is the degree of nilpotency of $T|_N$ we say that T is a *Kato type operator* of degree d.

Trivially the semi-regular operators, invertible operators for instance, and the quasinilpotent operators, nilpotent operators for instance, admit a generalized Kato decomposition. Now we turn to other classes of operators for which the Kato decomposition is not so trivial.

We say that $T \in B(X)$ is a *Fredholm operator* if

1. $\alpha(T) = \dim N(T) < \infty$,
2. $\beta(T) = \dim X/R(T) < \infty$

It is easy to see that every invertible operator is a Fredholm operator. Also, from (ii) follows that a Fredholm operator has closed range. Moreover, $R(T^n)$ is closed for any positive integer n and consequently $R^\infty(T)$ is the same. Let T_∞ be the restriction $T|_{R^\infty(T)}$, then T is Fredholm implies T_∞ is Fredholm.

The integer $i(T) = \alpha(T) - \beta(T)$ is called the index of the Fredholm operator T. An important result is the *Punctured Neighborhood Theorem*, which states that if $T \in B(X)$ is a Fredholm operator, then there exists

$\epsilon > 0$ such that $T+\lambda I$ is a Fredholm operator and $\alpha(T+\lambda I)$ and $\beta(T+\lambda I)$ are constant in the punctured neighborhood $0 < |\lambda| < \epsilon$. Furthermore,

$$\alpha(T+\lambda I) \leq \alpha(T), \quad \beta(T+\lambda I) \leq \beta(T), \quad i(T+\lambda I) = i(T), \quad \forall |\lambda| < \epsilon.$$

For the same ϵ, the *jump* $j(T)$ is defined as

$$j(T) = \alpha(T) - \alpha(T+\lambda I) = \beta(T) - \beta(T+\lambda I), \quad 0 < |\lambda| < \epsilon.$$

Thus, $j(T) \geq 0$ and the continuity of the index function i assures the second equality in the above equation. The jump of a Fredholm operator enables us to know when such an operator is semi-regular.

Theorem 6.3.1 ([2, Theorem 1.58]) *Let $T \in B(X)$ be a Fredholm operator. If $j(T) = 0$ then T is semi-regular.*

Proof. Since $j(T) = 0$, there exists $\epsilon > 0$ such that $\alpha(T+\lambda I)$ is constant for $|\lambda| < \epsilon$. Then

$$\alpha(T_\infty) \leq \alpha(T) = \alpha(T+\lambda I) = \alpha(T_\infty + \lambda I) \qquad \text{for every } 0 < |\lambda| < \epsilon.$$

Since T_∞ is Fredholm, using the Punctured Neighborhood Theorem we can choose $\epsilon > 0$ in such a way that $\alpha(T_\infty + \lambda I) \leq \alpha(T_\infty)$ for every $|\lambda| < \epsilon$. Thus, $\alpha(T_\alpha) = \alpha(T)$ so we have $N(T) \subseteq R^\infty(T)$.$\square$

In fact, the converse also holds, that is, if $T \in B(X)$ is Fredholm and semi-regular, then $j(T) = 0$. If $T \in B(X)$ is Fredholm of jump 0, trivially it admits a Kato decomposition. If it is not, the following lemma help us to construct such decomposition.

Lemma 6.3.1 ([2, Lemma 1.61]) *Let $T \in B(X)$ and suppose that $N^\infty(T) \nsubseteq R^\infty(T)$. Let $y \in X$ be such that*

$$y \in N(T^n) \text{ but } y \notin R(T).$$

Then

$$P = \sum_{j=0}^{n-1} T^{*j} f \otimes T^{n-j-1} y$$

is a bounded projection commuting with T. Furthermore, the range of P is the subspace Y generated by the elements $y, Ty, \ldots, T^{n-1}y$, the restriction $T|_Y$ is nilpotent and $j(T|_Y) = 1$.

T. Kato proved the following result in [10], which may be the reason for the name of the decomposition. Here we present the proof offered by P. Aiena.

Theorem 6.3.2 ([2, Theorem 1.62]) *Fredholm operators admit a Kato decomposition with* $\dim N < \infty$.

Proof. Let $T \in B(X)$ be a Fredholm operator. If T is semi-regular, taking $M = X$ and $N = \{0\}$ we get a Kato decomposition with $\dim N < \infty$.

Suppose now that T is not semi-regular. Hence $j(T) > 0$, so $N^\infty(T) \not\subseteq R^\infty(T)$. Let P be the projection in the above lemma. P commutes with T. The restriction $T|_{N(P)}$ is Fredholm and $j(T|_{N(P)}) = j(T) - 1$. Continuing with this process a finite number of times, the jump of the operator reduces to zero.□

We already know that if $T \in B(X)$ is Fredholm, then $T|_{R(T^n)}$ is Fredholm for every $n \in I\!N$. Now, if for some $n \in I\!N$ we have that $R(T^n)$ is closed and the restriction $T|_{R(T^n)}$ is Fredholm we say that T is a *B-Fredholm* operator.

If $T_n = T|_{R(T^n)}$ is Fredholm, we have

$$N(T_n) = N(T) \cap N(T^n) = N(T^{n+1})/N(T^n) \qquad \text{and}$$

$$X/R(T_n) = R(T^n)/R(T^{n+1}) = X/(R(T) + N(T^n))$$

(see[9], Lemma 3.2). Let us define the following quantities

$$\alpha_n = \dim \frac{N(T^{n+1})}{N(T^n)},$$

$$\beta_n = \dim \frac{R(T^n)}{R(T^{n+1})},$$

$$
\begin{aligned}
k_n &= \dim \frac{R(T^n) \cap N(T)}{R(T^{n+1}) \cap N(T)} = \dim \frac{R(T) + N(T^{n+1})}{R(T) + N(T^n)} \\
&= \alpha_n(T) - \alpha_{n+1}(T) = \beta_n(T) - \beta_n(T).
\end{aligned}
$$

Of course the differences in the last equation are valid whenever the involved quantities are finite. The interested reader can find proofs for the above equalities in the paper [9] and in chapter 3 of the book [16].

Note that $k_n(T) = 0$ if and only if $N(T) \cap R(T^n) \subset R(T^{n+1})$. Also $\alpha_0(T) = \alpha(T)$ and $\beta_0(T) = \beta(T)$. Furthermore, if T is B-Fredholm, $\alpha_n(T) < \infty$ and $\beta_n(T) < \infty$ for some $n \in \mathbb{N}$.

Now we will look for a decomposition into a Fredholm operator and a nilpotent one. First a lemma with some results we will need in the construction of our subspaces.

Lemma 6.3.2 ([16]) *Let $T \in B(X)$ be a B-Fredholm operator and let $n \in \mathbb{N}$ be the least integer such that $R(T^n)$ is closed and $T|_{R(T^n)}$ is Fredholm, then*

1. *$R(T^j)$ is closed for every $j \geq n$.*
2. *$N^\infty(T|_{R(T^n)}) = N^\infty(T) \cap R(T^n) \subset R^\infty(T)$.*
3. *$k_j(T) = 0$ for every $j \geq n$.*
4. *$T^* \in B(X^*)$ is B-Fredholm.*

Now we can construct a Kato type decomposition for B-Fredholm operators.

Theorem 6.3.3 ([16, Theorem 22.12]) *Let $T \in B(X)$. T is a B-Fredholm operator if and only if there exist T-invariant subspaces M, N such that $X = M \oplus N$, $T|_M$ is Fredholm and $T|_N$ is nilpotent.*

Proof. Let $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T|_{R(T^n)}$ is Fredholm. Then $\alpha_j(T) = \alpha_n(T) < \infty$ and $\beta_j(T) = \beta_n(T) < \infty$ for every $j \geq n$.

If $n = 0$, then T is Fredholm and the decomposition is trivial. So, suppose that $n \geq 1$.

Since $\dim R(T^n) \cap N(T) = \alpha_n(T) < \infty$, there exists a closed subspace L such that $X = L \oplus (R(T^n) \cap N(T))$.

Let us define inductively the closed subspaces N_j by $N_0 = \{0\}$ y $N_{j+1} = T^{-1}N_j \cap L$ $(j < n)$.

It is clear that $TN_{j+1} \subset N_j \cap R(T)$. Conversely, let $x \in N_j \cap R(T)$. Then there exists $u \in X$ such that $x = Tu$. Let us write $u = l + v$ with $l \in L$ and $v \in N(T) \cap R(T^n)$. Then $u - v = l \in L$ and $T(u - v) = Tu = x$. Thus, $u - v \in N_{j+1}$ and $x \in TN_{j+1}$.

Therefore

$$TN_{j+1} = N_j \cap R(T) \qquad (j < n).$$

We will prove by induction on j that $N_j \subset N_{j+1}$. The claim is clearly true for $j = 0$. Suppose that $j \geq 0$, $N_j \subset N_{j+1}$ and let $x \in N_{j+1}$. Then $Tx \in N_j \subset N_{j+1}$, so $x \in T^{-1}N_{j+1}$. Since $x \in N_{j+1} \subset L$, we conclude that $x \in N_{j+2}$.

Thus,

$$N_j \subset N_{j+1} \qquad (j = 0, 1, \ldots, n-1).$$

It is not hard to see that $N_j \subset N(T^j)$ for every j.

Now we will prove by induction on j that

$$N(T^j) \subset N_j + (N(T^j) \cap R(T^n)). \tag{6.1}$$

The inclusion is clearly true for $j = 0$. For $j = 1$ we have $N(T) = (N(T) \cap L) + (N(T) \cap R(T^n)) = N_1 + (N(T) \cap R(T^n))$. Let $j \geq 1$, $N(T^j) \subset N_j + (N(T^j) \cap R(T^n))$ and let $x \in N(T^{j+1})$. Then $Tx \in N(T^j)$, and hence $Tx = v_1 + v_2$ where $v_1 \in N_j$ and $v_2 \in N(T^j) \cap R(T^n) = N(T^j) \cap R(T^{n+1}) = T(N(T^{j+1}) \cap R(T^n))$. Thus $v_1 \in N_j \cap R(T) = T(N_{j+1})$ and

$$\begin{aligned} x &\in N_{j+1} + (N(T^{j+1}) \cap R(T^n)) + N(T) \\ &= N_{j+1} + (N(T^{j+1}) \cap R(T^n)) + (N(T) \cap L) + (N(T) \cap R(T^n)) \\ &= N_{j+1} + (N(T^{j+1}) \cap R(T^n)) + N(T). \end{aligned}$$

Therefore, (6.1) holds.

Finally, we will prove by induction that $N_j \cap R(t^n) = \{0\}$. It is clearly true for $j = 0$. Let $j \geq 0$, $N_j \cap R(T^n) = \{0\}$ and let $x \in N_{j+1} \cap R(T^n)$. Then

$Tx \in N_j \cap R(T^n)$ and hence, by the induction hypothesis, $Tx = 0$. Thus $x \in N(T) \cap R(T^n)$ and $x \in N_{j+1} \subset L$, so $x = 0$. Therefore

$$N_j \cap R(T^n) = \{0\}.$$

Let $N = N_n$. Then $TN \subset N$ and $N \subset N(T^n)$. Furthermore, $N + R(T^n) \supset N(T^n)$ by (6.1), and $N \cap R(T^n) = \{0\}$. Also, we have that $N + R(T^n) = N(T^n) + R(T^n) = T^{-n}R(T^{2n})$, which is closed since $R(T^{2n})$ is closed by the above lemma.

Now we will consider the dual operator $T^* \in B(X^*)$. We have that $R(T^{*j})$ is closed, $\alpha_j(T^*) = \beta_j(T)$ and $\beta_j(T^*) = \alpha_j(T)$ for every $j \geq n$. So we can use the same construction for T^*.

Since $\dim(N(T^*) \cap R(T^{*n})) = \alpha_n(T^*) = \beta_n(T) < \infty$, there exist a closed subspace $G \subset X$ of finite codimension such that ${}^\perp(N(T^*) \cap R(T^{*n})) \oplus G = X$. Let $L' = G^\perp$. Then L' is a closed subspace y the weak-* topology and $L' \oplus (N(T^*) \cap R(T^{*n})) = X^*$.

Let us define the subspaces $M'_0 \subset M'_1 \subset \ldots \subset M'_n = X^*$ by $M'_0 = \{0\}$ and $M'_{j+1} = T^{*-1}M_j \cap L'$. Using induction, M'_j is closed in the weak-* topology for every j.

Let $M' = M'_n$. As in the above construction, we have $T^*M' \subset M' \subset N(T^*n)$, $M' \cap R(T^{*n}) = \{0\}$ and $N(T^{*n}) \subset M' + R(T^{*n})$. Furthermore, $M + R(T^{*n})$ is a closed subspace.

Let $M = {}^\perp M'$. Then $T(M) \subset M$ and $M = {}^\perp M' \supset {}^\perp N(T^{*n}) = N(T^n)$.

Furthermore,

$$R(T^n) = {}^\perp N(T^{*n}) \supset {}^\perp(M' + R(T^{*n})) = {}^\perp M' \cap {}^\perp R(T^{*n}) = M \cap N(T^n)$$

and $M + N(T^n) = {}^\perp M' + {}^\perp R(T^{*n}) = {}^\perp(M' \cap R(T^{*n})) = X$.

Thus,

$$M + N \supset M + R(T^n) + N \supset M + R(T^n) + N(T^n) = X$$

and

$$M \cap N \subset M \cap N(T^n) \cap N \subset R(T^n) \cap N = \{0\}.$$

So we have $X = N \oplus M$, $T(N) \subset N$, $T(M) \subset M$ and $(T|_N)^n = O$.

Let $T_2 = T|_M$. We have

$$N(T_2) = N(T) \cap M \subset N(T^n) \cap M = {}^{\perp}(R(T^{*n} + M'))$$
$$= {}^{\perp}(R(T^{*n}) + N(T^{*n})) = N(T^n) \cap R(T^n) \subset R^{\infty}(T).$$

Thus $k_j(T_2) = 0$ for every $j \geq 0$. Hence the sequences $\alpha_j(T_2)$ and $\beta_j(T_2)$ are constant. Since $\alpha_n(T_2) = \alpha_n(T) < \infty$ and $\beta_n(T_2) = \beta_n(T) < \infty$, we conclude that $\alpha(T_2) < \infty$ y $\beta(T_2) < \infty$, and we get that T_2 is Fredholm.

Conversely, let $X = M \oplus N$, M, N be T-invariant subspaces and $T^n|_N = O$ and $T|_M$ Fredholm. Then $R(T^n) = R(T^n|_M)$ is of finite codimesion in M. Hence $R(T^n)$ is closed. It is not hard to very that $T|_{R(T^n)}$ is Fredholm.□

The class of operators admitting Kato decomposition can be characterized in terms of the range and kernel of the powers of an operator.

An operator $T \in B(X)$ is *quasi-Fredholm* if there exists $d \in I\!N$, called the degree of the quasi-Fredholm operator, such that $k_n(T) = 0$ for every $n \geq d$ and $R(T^{d+1})$ is closed.

Using Lemma 6.3.2 we see that B-Fredholm operators are quasi-Fredholm.

If $T \in B(X)$ is quasi-Fredholm, then $R(T) + N(T^d)$ and $N(T) \cap R(T^d)$ are closed. If they are also complemented, assumption that is automatically fulfilled if X is a Hilbert space, following the same steps of the proof of the above theorem we get the following.

Theorem 6.3.4 ([15, Theorem 5]) *Let $T \in B(X)$ be a quasi-Fredholm operator of degree d and suppose that the subspaces $R(T) + N(T^d)$ and $N(T) \cap R(T^d)$ are complemented. Then there exist T-invariant subspaces M, N such that $X = M \oplus N$, $T^d|N = O$ and $T|_M$ is semi-regular.*

It looks nicer in a Hilbert space setting.

Theorem 6.3.5 ([12, Theorem 3.2.1]) *Let H be a Hilbert space and $T \in B(H)$. Then T is quasi-Fredholm if and only if T admits a Kato decomposition.*

In view of our success characterizing the Kato decomposition, now we want to obtain something for the generalized Kato decomposition.

Let H be a Hilbert space. We say that an operator $T \in B(H)$ is *pseudo-Fredholm* if it admits a generalized Kato decomposition.

Unfortunately, for the pseudo-Fredholm operators only some technical results have been achieved and until now there are no "beautiful" characterizations in terms of subspaces of the range and kernel of an operator.

T. Kato proved his decomposition for Fredholm operators in the paper [10]. After that, J.P. Labrousse defined the quasi-Fredholm operators in [12] and proved that in a Hilbert space they were precisely those admitting a Kato decomposition. Later M. Mbekhta studied those operators admitting a generalized Kato decompositions, which he called pseudo-Fredholm in [14]. Then M. Berkani had the idea of investigating this theory paying attention to the restriction of an operator to the range of its powers, which led him to define in [4] the B-Fredholm operators. His work had a strong influence from the ideas of Labrousse and the research of C. Schmoeger with the generalized Fredholm operators (see [19]), which we have not discussed here because they turn out to be a special case of B-Fredholm operators. Schmoeger, in turn, based his work in a Caradus one ([6]). The proof of the Kato decomposition for B-Fredholm operators presented in this section is due to V. Müller, since Berkani proved it using a previous result of Labrousse whose proof was valid only for Hilbert spaces.

6.4 Fredholm theory through invertibility

Fredholm theory has an interesting relationship with invertibility.

Recall $K_0(X)$ is the set of finite rank operators and $K(X)$ is the set of compact operators. Let $\pi_0 : B(X) \to B(X)/K_0(X)$ and $\pi : B(X) \to B(X)/K(X)$ be the natural homomorphisms defined by $\pi_0(T) = T + K_0(X)$ and $\pi(T) = T + K(X)$.

Theorem 6.4.1 (Atkinson, see [16, Theorem 16.13]) *The following statements are equivalent:*

1. *$T \in B(X)$ is a Fredholm operator,*

2. *$\pi_0(T)$ is invertible in $B(X)/K_0(X)$*

3. *$\pi(T)$ is invertible in $B(X)/K(X)$.*

We should note that the quotient algebras $B(X)/K_0(X)$ and $B(X)/K(X)$ can be very different each other, for instance, the latter is Banach algebra while the former only got a seminorm, and even then both "give rise" to the same class of Fredholm operators.

Before proceeding to the discussion of the generalizations of Fredholm operators we are going to see in this section, we shall recall some generalized notions of invertibility.

We say that $T \in B(X)$ is *relatively regular* if there exist an operator $S \in B(X)$, called *generalized inverse*, such that

$$TST = T.$$

This operator S is also called an *inner inverse* for T. If $S = STS$ also holds for S we say that S is an *outer inverse* for T. The main issue with the relatively regular operators is that the generalized inverse is not unique: if S is a generalized inverse for T, taking $S' = STS$ we get that S' is an inner and outer inverse for T. For having uniqueness we need something more than inner and outer invertibility: $T \in B(X)$ is *group invertible* if there exists $S \in B(X)$ such that

$$TS = ST, \qquad STS = S \quad \text{and} \quad TST = T.$$

The group inverse is unique if it exists. It is known that T is group invertible if and only if $N(T) = N(T^2)$ and $R(T) = R(T^2)$. We should note that for a group inverse S of T we have $TST - T = T(I - ST) = 0$. Generalizing a little we have: $T \in B(X)$ is *Drazin invertible* if there exists $S \in B(X)$ such that

$$TS = ST, \qquad STS = S \quad \text{and} \quad T(I - ST) \text{ is nilpotent.}$$

The Drazin inverse is unique if it exists. It is known that T is Drazin invertible if and only if $N(T^n) = N(T^{n+1})$ and $R(T^n) = R(T^{n+1})$ for some $n \in \mathbb{N}$. If the conditions above hold for some n, then also hold for $n + 1$. The least

$n \in \mathbb{N}$ for which these conditions hold is called the Drazin index of T. Generalizing even more we have: $T \in B(X)$ is *generalized Drazin invertible* if there exists $S \in B(X)$ such that

$$TS = ST, \qquad STS = S \quad \text{and} \quad T(I - ST) \text{ is quasinilpotent.}$$

The generalized Drazin inverse is unique if it exists. It is known that T is generalized Drazin invertible if and only if $\{0\}$ is at worst an isolated point of $\sigma(T)$.

M.P. Drazin originally defined the inverse named after him in a more general context in a paper of 1958 ([7]). This inverse is a type of the so called spectral inverses, this means that if $S \in B(X)$ is Drazin inverse for $T \in B(X)$, then

$$\sigma(S) \setminus \{0\} = \{\lambda \in \mathbb{C} : \tfrac{1}{\lambda} \in \sigma(T) \setminus \{0\}\}.$$

For the generalizations we are going tu discuss here we will follow the idea of changing the invertibility in the Atkinson's theorem by generalized notions of invertibility.

We say that $T \in B(X)$ is a *generalized Fredholm operator* if $\pi_0(T)$ is group invertible in $B(X)/K_0(X)$.

The generalized Fredholm operators were originally defined, although in a different but equivalent way, by S.R. Caradus [6] and later were studied in a series of papers by C. Schmoeger [19]-[20]. Since the generalized Fredholm operators are a particular case of the B-Fredholm operators, they admit a Kato type decomposition.

Theorem 6.4.2 ([20, Theorem 1.1]) *$T \in B(X)$ is a generalized Fredholm operator if and only if there exist T-invariant closed subspaces $M, N \subseteq X$ such that $X = M \oplus N$, $T|_M$ is Fredholm and $T|_N$ is nilpotent with* $\dim R(T|_N) < \infty$.

Example 6.4.1 *A generalized Fredholm operator which is not Fredholm.*

Let $P \in B(X)$ be a projection such that $N(P)$ is not finite. We have that P is not Fredholm. Let $M = R(P)$ and $N = N(P)$. Hence, $P|_M$ is invertible and $P|_N = O|_N$, then, by the above theorem we have that P is a generalized Fredholm operator.

Generalizing a little more we get the B-Fredholm operators, that in this context we can call *Drazin-Fredholm*.

Theorem 6.4.3 ([5, Theorem 3.4]) *$T \in B(X)$ is a B-Fredholm operator if and only if $\pi_0(T)$ is Drazin invertible in $B(X)/K_0(X)$.*

Example 6.4.2 *A B-Fredholm operator which is not generalized Fredholm.*

Let $Q \in B(X)$ a nilpotent operator such that $R(Q)$ is not closed. Then, using $N = X$ in the above theorem we see that Q is B-Fredholm. Now, since $R(Q)$ is not closed, Q is not relatively regular. Then, using [19] (Proposition 2.1), for every $S \in B(X)$ we have that $QSQ - Q$ is not relatively regular. Since every finite rank operator is relatively regular, $QSQ - Q \notin K_0(X)$ for every $S \in B(X)$ and we get that $\pi_0(Q)$ is not relatively regular and therefore is not group invertible in $B(X)/K_0(X)$.

Note that until now we have used the quotient $B(X)/K_0(X)$. By Atkinson's theorem we can use $B(X)/K_0(X)$ or $B(X)/K(X)$ for studying Fredholm operators. This is no longer true for these generalizations. For example, if K is a compact operator with infinite dimensional range, then $\pi(K) = \pi(0)$ is group invertible in $B(X)/K(X)$ but $\pi_0(K)$ is not in $B(X)/K_0(X)$. Indeed, suppose $\pi_0(K)$ is group invertible, then by Theorem 6.4.2 it follows that if n is nilpotency degree of $K|_N$ then $R(K^n)$ is closed, which is a contradiction since K is compact and $R(K)$ is not finite dimensional implies $R(K^n)$ is not closed for every $n \in \mathbb{N}$.

We now turn to the generalized Drazin inverse case.

Definition 6.4.1 *An operator $T \in B(X)$ is* generalized Drazin-Fredholm *if $\pi(T)$ is generalized Drazin invertible in $B(X)/K(X)$.*

Example 6.4.3 *A generalized Drazin-Fredholm operator which is not B-Fredholm.*

Let T be the Volterra operator on the Banach space $X = C[0,1]$ defined by

$$(Tf)(t) = \int_0^t f(s)ds \qquad \textit{for all } f \in C[0,1] \textit{ y } t \in [0,1].$$

Then T is a quasinilpotent operator (but not nilpotent). Hence, 0 is an isolated point in $\sigma_{B(X)/K(X)}(\pi(T))$, so $\pi(T)$ is generalized Drazin invertible in $B(X)/K(X)$. Therefore, T is generalized Drazin-Fredholm.

Now note that if $T \in B(X)$ is B-Fredholm and n is the nilpotency degree of $T|_N$ of the Theorem 6.3.3 then $R(T^n)$ is closed. But it is known that, for the Volterra operator, $R(T^n)$ is not closed for every $n \in I\!N$, and therefore T is not B-Fredholm.

Note that we have defined the generalized Drazin-Fredholm operators using the ideal of compact operators $K(X)$. We will see later that this is equivalent to the use of the ideal $K_0(X)$.

Now we proceed to toward a Kato type decomposition for generalized Drazin-Fredholm operators. First we ask if these operators admit a generalized Kato decomposition.

Proposition 6.4.1 *If a compact operator K admits a generalized Kato decomposition, then $\sigma(K)$ is finite.*

Proof. Suppose that $K \in K(X)$ admits a generalized Kato decomposition, then by [2] (Theorem 1.41) we have that the analytical core $\mathcal{K}(T)$ is closed. Since K is a Riesz operator, using [13] (Corollary 9) we get that $\sigma(K)$ is finite.$\square$

Example 6.4.4 There exist generalized Drazin-Fredholm operators which does not admit a generalized Kato decomposition.

Suppose that $K \in K(X)$ is such that $\sigma(K)$ is not finite. From the definition of generalized Drazin invertibility it is clear that $\pi(K) = \pi(O)$ is generalized Drazin invertible in $B(X)/K(X)$. Hence, K is a Drazin-Fredholm operator, but by the above proposition we have that K does not admit a generalized Kato decomposition.

However, we have another decomposition. We say that T admits a *Fredholm-Riesz decomposition* if there exist two T-invariant closed subspaces $M, N \subset H$ such that $H = M \oplus N$ and the restriction $T|_M$ is Fredholm and $T|_N$ is Riesz.

Lemma 6.4.1 *If $\pi(T)$ is generalized Drazin invertible in $B(X)/K(X)$ then T admits a Fredholm-Riesz decomposition.*

Proof. Suppose that $\pi(T)$ is generalized Drazin invertible in $B(X)/K(X)$, then 0 is an isolated point in $\sigma_{B(X)/K(X)}(\pi(T))$. Let $\tilde{P}$ be the spectral idempotent for $\pi(T)$ corresponding to the spectral set $\{0\}$. If T is Fredholm or Riesz, then $\tilde{P}$ is $\pi(0)$ or $\pi(I)$ and the conclusion is trivial, so suppose that T is neither Fredholm nor Riesz. By [3] (Corollary 3.4 and Proposition 4.1) there exists a projection $P \in B(X)$ such that $\pi^{-1}(\tilde{P}) = P$. Let $M = N(P)$ and $N = R(P)$, it follows that $T|_M$ is Fredholm and $T|_N$ is Riesz.□

A.F. Ruston proved that $T \in B(X)$ is a Riesz operator if and only if $\pi(T)$ is quasinilpotent in the Banach algebra $B(X)/K(X)$ (see [17]). We will now prove that this is equivalent to be quasinilpotent in the ring $B(X)/K_0(X)$. Since the quotient $B(X)/K_0(X)$ is not a Banach algebra, the first thing we need is a suitable definition for quasinilpotent in rings.

As in [8], we say that an element q in a ring R is *quasinilpotent* if $1 + xq$ is invertible for every x such that $xq = qx$. It is not hard to see that in a Banach algebra this definition agrees with the usual definition.

In a ring R by R^{-1} we will denote the set of all invertible elements in R.

Lemma 6.4.2 *Let $T \in B(X)$. $\pi_0(T)$ is quasinilpotent in $B(X)/K_0(X)$ if and only if $\pi(T)$ is quasinilpotent in $B(X)/K(X)$.*

Proof. Suppose that $\pi_0(T)$ is quasinilpotent $B(X)/K_0(X)$, then for every $\pi(U) \in B(X)/K_0(X)$ such that $\pi(TU) = \pi(UT)$ we have that $\pi(I + UT) \in (B(X)/K_0(X))^{-1}$. Then, $\pi(I + \lambda T) \in (B(X)/K_0(X))^{-1}$ and we get that T is Riesz and therefore $\pi(T)$ is quasinilpotent in $B(X)/K(X)$.

Conversely, suppose that $\pi(T)$ is quasinilpotent in $B(X)/K(X)$. Let $U \in B(X)$ such that $UT - TU \in K_0(X) \subset K(X)$. We have that $\pi(TU) = \pi(UT)$, hence $\sigma_{B(X)/K(X)}\pi(TU) = \{0\}$ and we get that $\pi(I - TU)$ is invertible in $B(X)/K(X)$. Using the Atkinson's characterization (Theorem 6.4.1), $\pi_0(I - TU)$ is invertible in $B(X)/K_0(X)$. Hence, $\pi_0(T)$ is quasinilpotent in $B(X)/K_0(X)$.□

When considering the generalized Drazin inverse we can use both quotients indistinctly again.

Theorem 6.4.4 *Let $T \in B(X)$. The following statements are equivalent*

1. *$\pi_0(T)$ is generalized Drazin invertible in $B(X)/K_0(X)$.*

2. *$\pi(T)$ is generalized Drazin invertible in $B(X)/K(X)$.*

Proof. Suppose that $\pi_0(T)$ is generalized Drazin invertible in cociente algebra$B(X)/K_0(X)$, then there exist operators $S \in B(X)$, $K_1, K_2 \in K_0(X)$ such that

$$TS = ST + K_1, \qquad STS = S + K_2 \quad \text{and} \quad \pi(T(I - ST)) \text{ is quasinilpotent.}$$

Since $K_1, K_2 \in K(X)$ and using Lemma 6.4.2, we get that $\pi(T)$ is Drazin invertible in $B(X)/K(X)$.

Now suppose that $\pi(T)$ is generalized Drazin invertible in $B(X)/K(X)$. From Lemma 6.4.1, $H = M \oplus N$ and $T = T_1 \oplus T_2$, where $T_1 = T|_M$ is Fredholm and $T_2 = T|_N$ is Riesz. Since $T_1 : M \to M$ is Fredholm there exists $S_1 : M \to M$ such that $S_1T_1 = I + F_1$ and $T_1S_1 = I + F_2$ with $F_1, F_2 \in F_0(M)$. Let $S = S_1 \oplus O$. Then

$$ST = (T_1S_1 - F_2 + F_1) \oplus O = TS + ((F_1 - F_2) \oplus O),$$
$$STS = S_1(I + F_2) \oplus O = S + (S_1F_2 \oplus O) \quad \text{y}$$
$$T(I - ST) = T_1F_1 \oplus T_2 \quad \text{is Riesz.}$$

Therefore, $\pi_0(S)$ is the generalized Drazin inverse of $\pi_0(T)$ in $B(H)/F_0(H)$.□

Recall that for a Drazin-Fredholm (B-Fredholm) operator T we have that $R(T^n)$ is closed for some $n \in I\!N$. In the case of the generalized Drazin-Fredholm operators we loose this property.

Example 6.4.5 There exists an operator generalized Drazin-Fredholm K such that $R(K^n)$ is not closed for every $n \in I\!N$.

Let $X = \ell^2$ th space of the square-summable sequences, and let $K : \ell^2 \to \ell^2$ the operator defined by

$$K(x) = (0, x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \ldots), \qquad x = (x_1, x_2, x_3, x_4, \ldots) \in \ell^2.$$

Then K is compact and hence is generalized Drazin-Fredholm. Since K^n is compact and $R(K^n)$ is finite dimensional for every $n \in I\!N$, we get that $R(K^n)$ is not closed for every $n \in I\!N$.

6.5 Banach algebras

The set of all bounded linear operators $B(X)$ is a Banach algebra with the usual operations, taking the product as the composition of operators. This give us the idea of investigate to what extent we can extend the above results to general Banach algebras.

Let A be a complex Banach algebra with identity e. Most of the definitions are similar to those for $B(X)$. Of course that now the decompositions will be in an algebraic way using analogue to the projections.

An element $p \in A$ is *idempotent* if $p = p^2$ holds.

Let $P \in B(X)$ be a projection, then if $M = R(P)$ and $N = N(P)$ we have $X = M \oplus N$. Thus,

$$T + P = (TP + P)|_M \oplus (T + P)|_N = (T + I)|_M \oplus T|_N.$$

Note that if P is the spectral projection corresponding to the spectral set $\{0\}$ discussed in section 2, then $T|_M = TP$ is quasinilpotent, so $(T + I)|_M$ is invertible. Thus, $T+P$ is invertible. Conversely, if for an arbitrary projection P we have that $T + P$ is invertible and $TP = T|_M$ is quasinilpotent, we see that $T|_N$ is invertible.

In this section we will discuss the decomposition $a = ap + a(e-p)$, where ap and $a + p$ satisfy certain assumptions.

An element $a \in A$ is invertible if there exists $b \in A$ such that $ab = ba = e$. The spectrum of an element $a \in A$ is defined by

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda e \text{ is not invertible}\}.$$

An element $q \in A$ is *nilpotent* if there exists $n \in \mathbb{N}$ such that $q^n = 0$. We say that $q \in A$ is *quasinilpotent* if for every x commuting with q we have that $e - xa$ is invertible. Every nilpotent element is also quasinilpotent. This definition works in rings with identity and agrees with the definition usual definition in Banach algebras $\|q^n\|^{1/n} \to 0$.

Before stating a result on isolated points of the spectrum recall that the *spectral mapping theorem* assures us that if f is a holomorphic mapping in a neighborhood of $\sigma(a)$ then $\sigma(f(a)) = f(\sigma(a))$. By acc$\sigma(a)$ we will denote the set of accumulation points of $\sigma(a)$.

Theorem 6.5.1 ([11, Theorem 3.1]) *Let $a \in A$. Then $0 \notin \mathrm{acc}\sigma(a)$ if and only if there exists an idempotent $p \in A$ commuting with a such that ap is quasinilpotent and $a + p$ is invertible.*

Proof. If $0 \notin \sigma(a)$ then we can take $p = 0$. Hence, suppose that 0 is an isolated point of $\sigma(a)$. Then there exist open sets U and V of $\mathbb{C}$, such that $0 \in U$, $\sigma(a) \setminus \{0\} \subset V$, and $U \cap V = \emptyset$.

Let us define the function f in $U \cup V$ by $f(\lambda) = 0$ for $\lambda \in V$ and $f(\lambda) = 1$ for $\lambda \in U$. Then f is holomorphic in $\sigma(a)$ and $f(a) = p$ is the spectral idempotent of a corresponding to $\{0\}$. Moreover, p commutes with any b commuting with a.

Let $g(\lambda) = \lambda$ for $\lambda \in U \cup V$. Then $g(a) = a$. If $h(\lambda) = f(\lambda)g(\lambda)$, then $h(\lambda) = \lambda$ for $\lambda \in U$, and $h(\lambda) = 0$ for $\lambda \in V$. We get that $ap = h(a)$. Using the spectral mapping theorem we have $\sigma(ap) = \{0\}$, so ap is quasinilpotent. If $s(\lambda) = f(\lambda) + g(\lambda)$, then $s(\lambda) = \lambda + 1$ for $\lambda \in U$ and $s(\lambda) = \lambda$ for $\lambda \in V$. We see that $a + p = s(a)$. Again by the spectral mapping theorem we have that $0 \notin \sigma(a + p)$, so $a + p$ is invertible.

Conversely, if $p = 0$, then a is invertible and $0 \notin \sigma(a)$. Hence, suppose that there exist some idempotent $p \neq 0$ such that $ap = pa$, ap is quasinilpotent, and $a + p$ is invertible. For arbitrary $\lambda \in \mathbb{C}$ we have

$$\lambda - a = (\lambda - ap)p + (\lambda - (p + a))(1 - p).$$

Since $p+a$ is invertible, we have $0 \notin \sigma(p+a)$, so there exists $r > 0$ such that if $|\lambda| < r$ then $\lambda - (a+p)$ is invertible. Moreover, ap is quasinilpotent, hence $\lambda - ap$ is invertible for every $\lambda \in \mathbb{C} \setminus \{0\}$. Thus, for $0 < |\lambda| < r$ we have

$$(\lambda - a)^{-1} = (\lambda - ap)^{-1}p + (\lambda - (p + a))^{-1}(1 - p).$$

It follows that $0 \notin \mathrm{acc}\sigma(a)$. Let us use the same function f as above. Then $f(a)$ is the spectral idempotent of a corresponding to the spectral set $\{0\}$.

Let γ be a Cauchy contour separating $\{0\}$ from the set $\sigma(a) \setminus \{0\}$. We have

$$\begin{aligned} f(a) &= \frac{1}{2\pi i}\int_\gamma (\lambda - a)^{-1} d\lambda \\ &= \frac{1}{2\pi i}\int_\gamma (\lambda - ap)^{-1} p d\lambda + \frac{1}{2\pi i}\int_\gamma (\lambda - (p+a))^{-1}(e-p) d\lambda \\ &= \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} \lambda^{-n-1} a^n p d\lambda + 0 = p. \square \end{aligned}$$

Now we discuss the case in which $a + p$ is not invertible but some type of generalized invertible. We will use similar ideas to those in the previous section.

Let A be a Banach algebra. The radical of A is

$$\operatorname{rad}(A) = \{a \in A : 1 - xa \text{ is invertible for every } x \in A\}.$$

If $\operatorname{rad}(A) = \{0\}$ then we say that A is *semisimple*.

If A is semisimple, the sum of all minimal right ideals coincides with the sum of all minimal left ideals. This sum, which is also an ideal, will be called the *socle* of A, denoted $\operatorname{soc}(A)$.

We will not go into details about the socle and semisimple algebras, we will just recall that the algebra $B(X)$ is semsimple and the socle of $B(X)$ is the ideal of finite rank operators $K_0(X)$.

Let us denote by $\widehat{a} = a + \operatorname{soc}(A)$ the coset of a in $A/\operatorname{soc}(A)$.

An element $a \in A$ is *Fredholm* if $\widehat{a}$ is invertible in $A/\operatorname{soc}(A)$.

An element $a \in A$ is *group invertible* if there exists $b \in A$ such that

$$aba = a, \quad bab = b \quad \text{and} \quad ab = ba.$$

An element $a \in A$ is *generalized Fredholm* if $\widehat{a}$ is group invertible in $A/\operatorname{soc}(A)$.

Now we can have a decomposition like we have for B-Fredholm operators.

Theorem 6.5.2 *Let A be a semisimple Banach algebra. If $a \in A$ is a generalized Fredholm element, then there exists an idempotent $p \in A$ such that ap is nilpotent and $a + p$ is Fredholm.*

Proof. Suppose $a \in A$ is generalized Fredholm. Then, there exists $b \in A$ such that

$$\widehat{a}\widehat{b}\widehat{a} = \widehat{a}, \quad \widehat{b}\widehat{a}\widehat{b} = \widehat{b} \quad \text{y} \quad \widehat{a}\widehat{b} = \widehat{b}\widehat{a}.$$

Since $aba - a \in \mathrm{soc}(A)$, and the elements of $\mathrm{soc}(A)$ are relatively regular, $aba - a$ has generalized inverse, say r. Hence,

$$b_0 = b - r + bar + rab - barab$$

is a generalized inverse for a. Also

$$\begin{aligned}\widehat{a}\widehat{b_0} &= \widehat{a}\widehat{b} - \widehat{a}\widehat{r} + \widehat{a}\widehat{b}\widehat{a}\widehat{r} + \widehat{a}\widehat{r}\widehat{a}\widehat{b} - \widehat{a}\widehat{b}\widehat{a}\widehat{r}\widehat{a}\widehat{b}\\ &= \widehat{a}\widehat{b} - \widehat{a}\widehat{r} + \widehat{a}\widehat{r} + \widehat{a}\widehat{r}\widehat{a}\widehat{b} - \widehat{a}\widehat{r}\widehat{a}\widehat{b}\\ &= \widehat{a}\widehat{b}.\end{aligned}$$

In a similar way we see that $\widehat{b_0}\widehat{a} = \widehat{b}\widehat{a}$. Now, since $ab_0a = a$, we have that $p = e - b_0a$ is idempotent. Hence

$$\widehat{a} + \widehat{p} = \widehat{a} + \widehat{e} - \widehat{b_0}\widehat{a} = \widehat{a} + \widehat{e} - \widehat{b}\widehat{a},$$

and $(\widehat{a} + \widehat{e} - \widehat{a}\widehat{b})(\widehat{b}\widehat{a}\widehat{b} + \widehat{e} - \widehat{a}\widehat{b}) = \widehat{e}$, so $\widehat{a} + \widehat{p}$ is invertible and $a + p$ is Fredholm. Also,

$$ap = a(e - b_0a) = a - ab_0a = a - a = 0,$$

so ap is nilpotent.□

6.6 References

[1] M. Ahues, A. Larguillier, B. V. Llimaye, Spectral Computations for Bounded Operators. Chapman & Hall/ CRC, Boca Raton, 2001.

[2] P. Aiena, Fredholm and Local Spectral Theory with Applications to Multipliers. Kluwer Academic Publishers, Dordrecht, 2004.

[3] B. Aupetit, E. Makai Jr., M. Mbekhta, J. Zemánek, *The connected components of the idempotents in the Calkin algebra, and their liftings*, Operator theory and Banach algebras (Rabat, 1999), Theta, Bucharest, 2003, 23–30.

[4] M. Berkani, *On a class of quasi-Fredholm operators,* Integral Equations Oper. Theory **34**, (1999), 244–249.

[5] M. Berkani, M. Sarih, *An Atkinson-type theorem for B-Fredholm operators,* Studia Math. **148**, (2001), 251–257.

[6] S.R. Caradus, *Operator theory of the pseudo-inverse*, Queen's papers in pure and appl. math. **50** (1978).

[7] M.P. Drazin, *Pseudo-inverses in associative rings and semigroups*,

Amer. Math. Monthly, **65**, (1958), 506–514.

[8] R. Harte, *On quasinilpotents in rings*, Panamer. Math. J. **1** (1991), 10–16.

[9] M.A. Kaashoek, *Ascent, Descent, Nullity and Defect, a Note on a paper by A.E. Taylor,* Math. Annalen **172**, (1967), 105–115.

[10] T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators*, Journal d'Analyse Mathématique, **6** (1) (1958), 261–322.

[11] J.J. Koliha, *A generalized Drazin inverse,* Glasgow Math. J. **38**, (1996), 367–381.

[12] J.P. Labrousse, *Les operateurs quasi Fredholm: une generalization des operateurs semi Fredholm,* Rend. Circ. Mat. Palermo **29**, (1980), 161–258.

[13] T. Len Miller, V.G. Miller, M.M. Neumann, *On operators with closed analytic core*, Rend. Circ. Mat. Palermo, **II, LI** (2002), 495–502.

[14] M. Mbekhta, *Operateurs Pseudo-Freedholm. I: Resolvant Generalise*, Journal of Operator Theory, **24**(2) (1990), 255–276.

[15] V. Müller, *On the Kato decompositon of quasi-Fredholm and B-Fredholm operators,* Geometry in Functional Analysis, Erwin Schrodinger Institute, Wien 2000.

[16] V. Müller, Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras. Birkhauser Verlag, Basel, 2003.

[17] A.F. Ruston, *Operators with a Fredholm theory*, Journal of the London Mathematical Society, **29**, 1954, 318–326.

[18] B.P. Rynne, M.A. Youngson, Linear Functional Analysis. Springer-Verlag, London, 2008.

[19] Ch. Schmoeger, *On a class of generalized Fredholm operators, I* Demonstratio Mat., **XXX** 4, (1997), 829–842.

[20] C. Schmoeger, *On a class of generalized Fredholm operators, V*, Demonstratio Math., **32** 5, (1999), 595-604.

7

Generalized uniform weighted approximation of continuous functions from Haar spaces

Miguel Antonio Jiménez-Pozo, Ivonne L. Martínez-Cortés, José Nobel Méndez-Alcocer and Victor Manuel Méndez-Salinas [1]

Abstract: Given a real valued function f and non-negative real valued functions h_1 and h_2 on the same domain, we construct neighborhoods centered at f and borders $f - \lambda h_1$ and $f + \lambda h_2$, with $\lambda \geq 0$. We measure the approximation to f from a given $n-$dimensional Haar space by minimizing the amplitude λ of the neighborhoods while they contain elements of the approximating functions. We discuss several recent results on this subject, in particular uniqueness of the best approximation by means of a generalized Chebyshev alternation theorem and Chebotarev criterion. The characterization of the best generalized polynomial of approximation leads to an extended Remez algorithm. Thus using this algorithm we present examples of approximation.

[1]Facultad de Ciencias Físico Matemáticas, Benemérita Universidad Autónoma de Puebla, Ave. San Claudio y 18 Sur, C. U. de San Manuel, Puebla, Pue.72570 , México., mjimenez@fcfm.buap.mx

Keywords: Best approximation, Chebyshev alternation theorem, Chebotarev criterion, Haar systems, varying amplitude, sign sensitive weight, asymmetric norm.

Classification: MSC 41A50, 41A29 and 41A52.

7.1 Introduction

This manuscript is planned to introduce the ideas involved in the generalized uniform weighted approximation. Several important results are given and the proofs of them are included or outlined. References have been quoted.

As known, the simplest but most important problem in Approximation Theory is to find an element from an approximating class $\mathcal{C}$ of functions to a given function f. The most typical class $\mathcal{C}$ is a set of algebraic or trigonometric polynomials, and the tools of measuring the approximation are the uniform norm or the least squares technique. As in many mathematical problems, firstly one needs to analyze the existence of a solution, i. e., to know whether there exists p in $\mathcal{C}$, called an element of best approximation relative to the problem in hands, such that p is the nearest one in $\mathcal{C}$ to f. In case of a positive answer, to search for the uniqueness of elements of best approximation. Finally a method is needed for calculating p and its distance to f, this distance is called the best approximation to f from the approximating class $\mathcal{C}$.

The popularity of the least square method in this subject is not by hazard. This way of measuring corresponds to a metric associated to a scalar product, a concept which not only is used for modeling many physical and technical problems, but whose theory is very rich and comfortable. In fact, in dealing with any finite dimensional space $\mathcal{C}$, we not only obtain positive answers to the problem of existence and uniqueness, but a feasible way of calculus by means of the standard techniques of optimization or by Fourier orthogonal expansions.

In general the use of one or another approximating class of functions and the way of measuring the approximation depend on the practical problem in hands, and uniform approximation is very often required. The theory of approximation with uniform norm is large and a very extensive collection of results on the subject can be easily found in text books and research papers. However mathematical difficulties may appear if the elements in the approximating class should satisfy constrains. This is exactly the case we shall examine in this paper. In fact, different situations lead us to consider the following setting:

Problem : "We consider a real-valued target function f on a certain domain D, a class $\mathcal{C}$ of possible approximating functions, non-negative (called *errors*) functions h_1 and h_2 to bound the errors of measurement, and the problem of choosing an approximating element $p \in \mathcal{C}$ to f and a real $\lambda \geq 0$ such that the inequalities

$$-\lambda h_1 \leq p - f \leq \lambda h_2$$

are uniformly satisfied on the domain D.

The infimum of these λ (if it has a sense), may be defined as the best uniform approximation to f from the class $\mathcal{C}$, relatively to the given error functions. The case $h_1 = h_2 = 1$ coincides with the uniform (also call Chebyshev) approximation mentioned above. While $h_i = 1$ and $h_j = 0$, $i \neq j$, represents a one-sided uniform approximation problem. Thus a variety of different cases are included in this general setting. Consequently, the answers we could obtain to the possible mathematical questions strongly depend on the choice of the data D, C, the error functions, etc."

This subject has been early treated and focused in different ways by mathematicians from Western and Eastern countries. However it is rather rare to find crossed references between mathematicians from both geographical areas. Thus several known but introductory results we need are presented in next Section 2, wherein we shall also quote the main references. In Section 3, we study the uniqueness of best approximation for the problem developed below. Finally in Section 4 we shall include examples of approximating polynomials obtained by a generalized Rémez algorithm to show differences with the traditional uniform polynomial approximation.

7.2 Chebyshev asymmetric approximation by generalized polynomials.

We begin by defining the approximating class $\mathcal{C}$ to be employed in this paper. Let $g_{1,}\ g_2, ..., g_n \in C\,[a,b]$ be a *Haar* (also called *Chebyshev*) system. This means that for any set of n different points

$$a \leq x_1 < x_2 < ... < x_n \leq b$$

the determinant

$$|g_i\,(x_j)|_{1\leq i,j\leq n}$$

is not zero. A *Haar* space is a subspace $\mathcal{C}$ of $C\,[a,b]$, generated by any Haar system. Its elements are named *generalized polynomials.* This name surely comes from the fact the most usual Haar systems are the algebraic polynomials generated by the Haar system $1, x, \cdots, x^m$, with dimension $n = m+1$, or the trigonometric polynomials generated by the Haar system 1, $\cos x$, $\sin x, \cdots, \cos mx$, $\sin mx$, with dimension $n = 2m+1$. Non trivial Haar spaces are possible only on topological spaces homeomorphic to real intervals or to the circle T of complex numbers $z, |z| = 1$, with the topology associated to the Euclidean distance. In particular T with the boundedly equivalent distance defined between any two points by the length of the shortest arc joining these points, is isometrically isomorphic to $[0, 2\pi)$ with the distance

$$d(x,y) := min\{|x-y|, 2\pi - |x-y|\}.$$

Thus for simplification, we can consider only real intervals $[a,b]$, with the identification of $-\pi$ and π, if periodic functions are considered on T. The standard reference here could be [14].

Now we precise the problem to be analyzed in this paper. Following ($Problem$) in Section 1, let D be a compact set of the real line with at least $n+1$ points (usually a whole interval $[a,b]$, but we shall accept this general setting). The real-valued functions f , h_1, h_2 are considered to be continuous on D, where h_1, h_2 are positive but one of them is strictly positive The case in which h_1, h_2 have common zeros has been recently considered in [16] for treating interpolation problems, but this case exceeds the objectives of this introductory monograph. Suppose $[a, b]$ is the convex hull of the set D. Then $\mathcal{C}$ will be any n-dimensional Haar space of continuos functions on $[a, b]$. For each $\lambda \geq 0$ we define the sets $B_\lambda = B_\lambda(f, h_1, h_2)$ by

$$B_\lambda = \{g \in \mathcal{C}[D] : \forall x \in D, \ (f - \lambda h_1)(x) \leq g(x) \leq (f + \lambda h_2)(x)\}, \quad (7.1)$$

and

$$M_\lambda = M_\lambda(f, h_1, h_2) = \mathcal{C} \cap B_\lambda. \quad (7.2)$$

Using a non rigorous but suggestive language we may say that B_λ is a band with amplitude λ, inferior and superior borders $f - \lambda h_1$ and $f + \lambda h_2$ respectively, and that p is within the given band whenever belongs to M_λ. Moreover, that p touches a border whenever $p \in M_\lambda$ and one of the inequalities in (7.1) becomes an equality for $g = p$ at some $x \in D$. The set B_λ grows when the parameter λ increases. Then, following a terminology used in the theory of topological vector spaces, the main point is that the set M_λ absorbs all approximating elements when the parameter λ increases. By the way, this is not true if h_1 and h_2 have common zeros.

Definition 7.2.1 *The best approximation* $E(f) = E(f, h_1, h_2)$ *to* f *from the class* $\mathcal{C}$, *with respect to the error functions* h_1 *and* h_2, *is defined by*

$$inf\{\lambda \geq 0 : M_\lambda(f, h_1, h_2) \neq \phi\}. \quad (7.3)$$

Any generalized polynomial $p \in \mathcal{C}$ *is of best approximation in this context whenever* p *is within the band of amplitude* $E(f)$.

Taking in consideration the geometrical meaning of this technique of measuring the distance between p and f, it was denominated *varying amplitude method of approximation* in [12], [13] and [16]. Now we are looking for answers to the typical questions of existence of polynomials of best approximation, uniqueness, and so on. In the particular case described above we claim:

Theorem 7.2.1 (Existence)
i) The best approximation $E(f)$ *is attained.*
ii) There always exists a polynomial of best approximation.

Proof. Since one of the error functions is strictly positive and we are dealing with continuous functions on compact domains, the sets M_λ in (7.2) are non-empty for large values of λ. Observe that M_λ in (7.2) is closed and bounded. Since $\mathcal{C}$ is a finite dimensional the sets M_λ are compact. Then by applying a compactness argument we deduce the existence of a minimum value $\lambda = E(f)$ such that $M_\lambda(f, h_1, h_2) \neq \phi$. To prove the second part, take any p in $M_{E(f)}$.□

Of course the proof above may be extended to more general situations. Now the new problem is to examine uniqueness of the generalized polynomial of best approximation, a task that will be accomplished in the next section. But first some comments will be convenient and also necessary for recognizing the original works.

Define $w_i = 1/h(i)$, $i = 1, 2$, with the agreement that $1/0 = \infty$ but $0 * \infty = 0$. For any real valued function g set $g^+ = (|g| + g)/2$ and $g^- = (|g| - g)/2$. Observe that

$$E(f) = \underset{p\in\mathcal{C}}{inf}\ \underset{p\in D}{sup}\left\{\left((p-f)^- w_1 + (p-f)^+ w_2\right)(x)\right\}. \tag{7.4}$$

Thus the varying amplitude method of approximation is a special kind of uniform weighted approximation. Observe that here the weight is given by the two functions w_1 and w_2, that in the general case may be different. They act on the negative and positive part of f, respectively. Due to this reason this pair of functions was called *sign sensitive weight* by Dolzhenko and Sevastyanov in their survey paper (see [8]) and the long list of references quoted there). Sign sensitive weight is an evolution of the ideas presented by Krein and Nudelman in 1973 in their book [15], where strictly positive continuous error functions are considered in (7.4). However to our knowledge, the first and indeed more general weight of this kind is due to Moursund [20] so early as in 1966 and continued in subsequence papers (see the survey [3]).

It is important to observe that generally speaking $E(f) \neq E(-f)$. We may study approximation problems by substituting norms by *asymmetric* norms or more general by positive homogeneous functionals. Such a functional is a positive real or infinite valued function ρ defined on a cone $\mathbb{C}$ of a real vector space $\mathbb{E}$, that satisfies the properties

$$(i)\ \ \forall f, g \in C, \rho(f+g) \leq \rho(f) + \rho(g)$$

$$(ii)\ \ \forall f \in C \forall \alpha \geq 0, \rho(\alpha f) = \alpha \rho(f)\,.$$

But perhaps there exists $f \in C$, such that $-f \in C$ and $\rho(f) \neq \rho(-f)$, or such that $f \neq 0$ and $\rho(f) = 0$. An asymmetric norm is a positive valued homogeneous functional ρ defined on the whole space E that must satisfy together with (i) and (ii) the properties

$$(iii)\ \ (\rho(f) = 0) \Leftrightarrow (f = 0)$$

$$(iv)\ \ \exists M > 0 \forall f \in E, \rho(-f) \leq M\rho(f)\,.$$

If we define on $C[D]$ the functional

$$\rho(g) = \underset{p \in D}{sup} \left\{ \left(g^{-} w_1 + g^{+} w_2 \right)(x) \right\} \tag{7.5}$$

and measure the approximation between two functions p and f by $\rho(p-f)$ with just this order, it follows that ρ in (7.5) is a positive homogeneous functional that becomes an asymmetric norm if both of the error functions are strictly positive. Measuring the best approximation to f from elements $p \in \mathcal{C}$ by the traditional formula

$$E(f) = \underset{p \in \mathcal{C}}{inf} \left\{ \rho(p-f) \right\},$$

we get the same number given by (7.4). Thus all reviewed methods here are equivalent.

The theory of asymmetric normed space has found application not only on approximation theory but also on computer science. To quote some references see [1], [2], and [17].

7.3 Uniqueness of the best approximation

In a relatively recent paper, Pokrovskii [22] studied the uniqueness problem of best approximation by positive homogeneous functionals in an exhaustive collection of different examples. In the particular context with wich we are dealing, Moursund in [20] and later Krein and Nudelman in [15] have established the uniqueness of this approximation process by means of a general Chebyshev Alternation Theorem.

Concerning Chebyshev theorem [6] some interesting commentaries and general ideas behind the subject can be found in [25]. Moreover this theorem may appear unexpectedly without any apparent connection to approximation problems. In fact, verifying the hypothesis of the Karush-Kuhn-Tucker theorem for semi-infinite optimization [24] in a problem arising from the petroleum industry, Guerra and Jimenez [9] found an extreme case in which a certain constraint qualification is violated, but the theorem applies. This situation corresponds to the case in which the feasible set is reduced to a singleton. Then they were brought to announce and prove independently the general Chebyshev alternation theorem in hands ([10] and [11]).

While Chebyshev theorem is stated in connection with a class of approximating functions in a linear space $\mathcal{C}$, Pokrovskii came back to an old but not well known criterion of characterization of the best uniform approximation that was stated by Chebotarev in [4] and [5], and could be used in dealing with approximating functions defined on a convex set $\mathcal{C}$. The substitution of linear spaces by only convex sets is not so important in connection with the approximation from Haar linear spaces, but very important in dealing with other classes $\mathcal{C}$, such as rational functions. Probably the reason for which Chebotarev criterion is not so well known is because it also promoted further developments which finally conducted to the useful Kolmogorov criterion (see the classical criterion in [7], for instance).

There is however an equivalence between Chebyshev and Chebotarev criteria for linear spaces $\mathcal{C}$. In fact, in the classical case both criteria are equivalent to the condition that a certain convex set in the euclidean $n+1$ dimensional space contains the zero vector as an interior point. But this last property is known to be true in the general Chebyshev Alternation Theorem [10] and [11]. So, since the generalization of Chebotarev theorem to weighted uniform approximation conserves this last property [18], we arrived at the equivalence between both concepts in the general case we are dealing with.

Everything that we have explained here about both criteria is the content of the next very general theorem for which we need several definitions.

For the moment, we suppose $D = [a, b]$.

Let $f, h_1, h_2 \in C[a, b]$ be such that $h_1, h_2 > 0$, $g_1, \ldots, g_n \in C[a, b]$ a Haar system; $E = E\,(f)$ the best approximation from the linear space $\mathcal{C}$ generated by this Haar system to f relative to the errors h_1 and h_2.

(i) For each $p \in \mathcal{C}$, define the sets:

$$R(p, h_1) := \{x \in [a, b] | (p - f + Eh_1)(x) = 0\},$$

$$R(p, h_2) := \{x \in [a, b] | (p - f - Eh_2)(x) = 0\},$$

$$R(p) := R(p, h_1) \cup R(p, h_2).$$

(ii) For $h \in C[a, b]$ and $x \in [a, b]$, also define the vectorial function

$$a(h, x) := sign(h(x))(g_1(x), g_2(x), \ldots, g_n(x)).$$

Theorem 7.3.1 (Characterization) *Let $g_1, \ldots, g_n \in C[a, b]$ be a Haar system that generate the approximating space $\mathcal{C}$, and $f, h_1, h_2 \in C[a, b]$, $h_1, h_2 > 0$, $f \notin \mathcal{C}$, $E\,(f)$ the best approximation from $\mathcal{C}$ to f. Then the following statements are equivalent:*

a) $p \in M_{E(f)}$, i.e. p is a polynomial of best approximation from $\mathcal{C}$ to f with respect to h_1 and h_2.

b) There exist $n + 1$ different points $z_1 < \cdots < z_{n+1}$ in $[a, b]$, such that p takes the alternation values $f(z_i) + \lambda_n h_2(z_i)$ and $f(z_i) - \lambda_n h_1(z_i)$ (or $f(z_i) - \lambda_n h_1(z_i)$ and $f(z_i) + \lambda_n h_2(z_i)$) for $i = 1, 3, \cdots$, while possible.

c) $R(p)$ has $n + 1$ elements

$$x_1 < x_2 < \cdots < x_{n+1},$$

such that the zero vector of $\mathbb{R}^n$ *belongs to the interior of the convex hull of the vectors*

$$\{a(p-f,x_1),\cdots,a(p-f,x_{n+1})\}.$$

d) For every $c\in\mathbb{R}^n\backslash\{0\}$*, there exist* $x_c,y_c\in R(p)$ *such that*

$$a(p-f,x_c)\cdot c>0 \quad and \quad a(p-f,y_c)\cdot c<0$$

e) For every $q\in\mathcal{C}$*,* $q\neq 0$

$$\underset{p\in R(p)}{max}(p(x)-f(x))(q(x))>0.$$

Remark 7.3.1 Chebyshev alternation theorem is given in (b), Chebotarev criterion in (d), and Kolmorogov criterion in (e).

Proof. The complete proof may be found in [18].□

Theorem 7.3.2 (Uniqueness) *The polynomial* p *of best approximation from* $\mathcal{C}$ *to* f *with respect to* h_1 *and* h_2 *is unique.*

Proof. In dealing with algebraic polynomials, for instance, we have at hand many results about multiplicities of zeros. In general Haar spaces differentiation cannot be assumed and the same theory does not apply. But a case of double zero has been studied in [7] for functions in Haar spaces. With that study in hands, suppose $p,q\in M_{E(f)}$. Using Chebyshev alternation theorem we prove $p-q$ has $n+1$ isolated zeros in $[a,b]$ with only a possible exception of one double zero. Then $p-q=0$.□

7.4 Extensions and applications

Now suppose $D\subsetneq[a,b]$ is not excluded.

Theorem 7.4.1 (Chebyshev alternation theorem generalized) *Any polynomial* $p\in\mathcal{C}$ *of best approximation related to the functions* f*,* h_1 *and* h_2*, satisfies the following generalized alternation property:*

"*There exist* $n+1$ *points* $z_i \in D$, $z_1 < z_2 < .. < z_{n+1}$, *such that*

$$p(z_i) = f(z_i) + E(f)\,h(z_i) \;\; \textit{and} \; p(z_{i+1}) = f(z_{i+1}) - E(f)\,h(z_{i+1})$$

or

$$p(z_i) = f(z_i) - E(f)\,h(z_i) \;\; \textit{and} \; p(z_{i+1}) = f(z_{i+1}) + E(f)\,h(z_{i+1}),$$

for $i = 1, 2, \cdots, n$."

Proof. It may be found in [10] or [11].□

Remark 7.4.1 Anyway we shall explain the main steps of the proof. If $D = [a, b]$, we prove the theorem following the traditional cannons. If $D \subsetneq [a, b]$, a theorem of Cantor asserts the complement of D in $[a, b]$ can be represented as a countable union of disjoint relative open intervals (I_n). Since $[a, b]$ is the convex hull of D, the extreme elements a, b belong to D. Thus each $I_n = (a_n, b_n)$ is indeed an usual open interval. Further, we reduce the proof for general compact domains with at least $n+1$ elements to the known case $D = [a, b]$ by extending the functions f, h_1 and h_2 to the whole interval. To doing it, we inductively define f on each I_n by joining $(a_n, f(a_n))$ and $(b_n, f(b_n))$ in an affine way. We also define affine extensions of h_1 and h_2, by joining $(a_n, h_i(a_n))$ with $((a_n + b_n)/2, c_n)$ and $((a_n + b_n)/2, c_n)$ with $(b_n, h_i(b_n))$, where c_n is chosen so large that the polynomial of best approximation to the extended functions cannot touch the borders of the band in any point within the intervals I_n.

Corollary 7.4.1 *The polynomial of best approximation related to the functions* f, h_1 *and* h_2 *on general compact domains is unique.*

Proof. It follows the typical scheme of the previous section.□

As an application of this generalized *Chebyshev alternation theorem,* a version of the well known Remez algorithm [23] can be developed following the traditional cannons (see [22] [16] for algebraic polynomials and [19] for trigonometric ones). To finish, using such an algorithm, we present the examples below where the differences between the traditional uniform approximation and the general weighted uniform approximation may be clearly observed. The figures that illustrate the examples are given at the end of the paper as an annex.

Suppose the periodical case in which the circle group T of complex numbers z, $|z| = 1$, is taken to be the interval $[-\pi, \pi]$, with the identification between -π and π. We shall approximate the W-function f defined by parts

$$f(x) = \left\{ \begin{array}{ll} -\frac{2}{\pi}x - 1, & x \in \left[-\pi, -\frac{\pi}{2}\right] \\ \frac{2}{\pi}x + 1, & x \in \left[-\frac{\pi}{2}, 0\right] \\ -\frac{2}{\pi}x + 1, & x \in \left[0, \frac{\pi}{2}\right] \\ \frac{2}{\pi}x - 1, & x \in \left[\frac{\pi}{2}, \pi\right] \end{array} \right\},$$

by a trigonometric polynomial $p(x) = a_0 + a_1 \cos x + b_1 \sin x$ in the following cases:

1. Error functions $h_1{=}h_2 = 1$, i. e. the traditional uniform approximation.

Applying Chebyshev theorem we find by simple inspection that the constant function $p_1(x) = 0.5$ is the polynomial of best approximation, $E(f) = 0.5$, and contact points of the polynomial of best approximation with the borders of the band are located at $-\pi, -\pi/2, 0$, and $\pi/2$.

2. Error functions $h_1 = 2 + \cos x$, $h_2 = 2 - \cos 3x$. Since all involved functions are even, the polynomial of best approximation p_2 must be an even function too. Running an addapted version of Remez algorithm to this general way of approximation, we obtain $p_2 \simeq 0.44 - 0.28\text{cos}x$, $E(f) = 0.28$, and that contact points of the polynomial of best approximation with the borders of the band are located at -$\pi, -1.94, 0$, and 1.94.

3. Error functions $h_1 = 2 + \cos x$, $h_2 = 2 - \sin 3x$. Although the target function is even, one of the error function is not. The calculated polynomial of best approximation is $p_3 \simeq 0.49 - 0.26 \cos x + 0.23 \sin x$, $E(f) = 0.26$, and that contact points of the polynomial of best approximation with the borders of the band are located at $-\pi, -\pi/2, 0$, and 2.32.

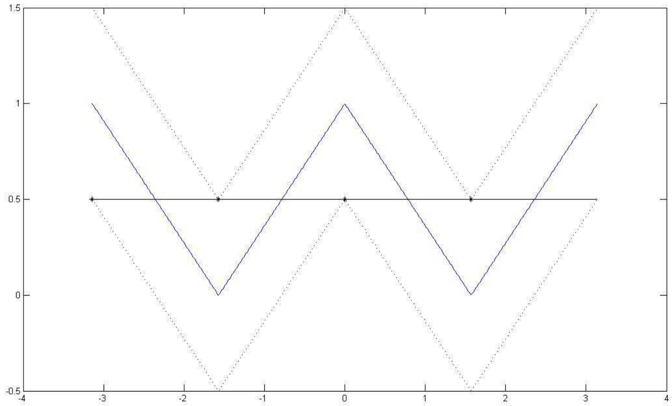


Figure 7.1:

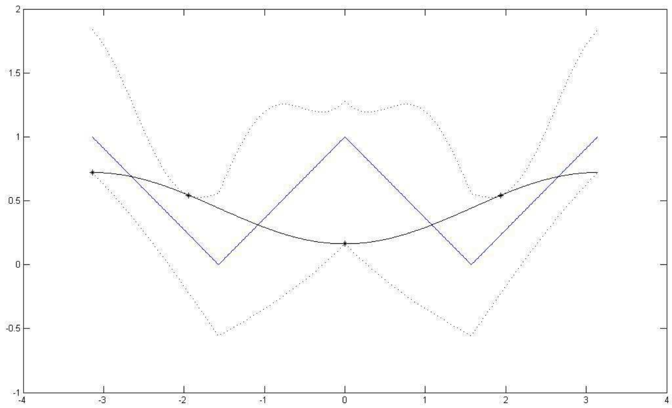


Figure 7.2:

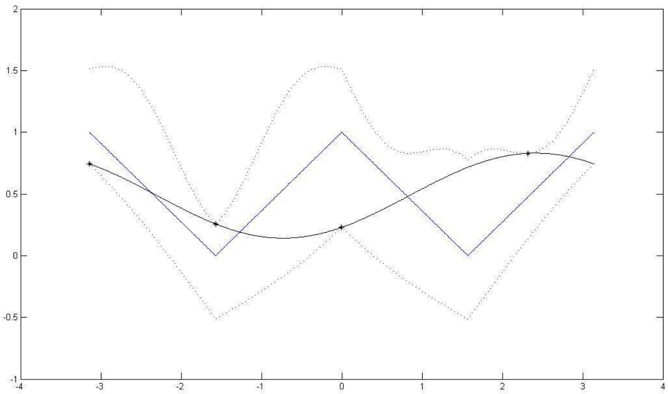


Figure 7.3:

7.5 References

[1] Alegre C., I. Ferrando, L. M. García, E. A. Sánchez-Pérez, *Compactness in asymmetric normed spaces,* Topology Appl., 155 (2008) 527-539.

[2] Alegre , C., *Continuous operators on asymmetric normed spaces,* Acta Mathematica Hungarica, Vol. 122, 4 (2009) 357-372.

[3] Chalmers, B. L., G. D. Taylor, *Uniform approximation with constraints.* Jber. d. Math.-Verein. 81, 2 (1979) 49-86.

[4] Chebotarev, N. G., *On general minimax criterion.* Dokl. Akad. Nauk SSSR, 39-1, (1943), 373-376 (Russian).

[5] Chebotarev, N. G., *A minimax criterion and its applications*, *Collected papers* Vol 2, Akad. Nauk SSSR, Moscow-Leningrad (1949) 396-409, (Russian).

[6] Chebyshev, P. L., *Sur les questions de minima qui se rattachent à la représentation approximative des fontions.* Mém. Prés. Acad. Imp. Sci. Pétersb. (6), Sci. Math. Phys. VII, (1859), 199-291.

[7] DeVore, R. A., G. G. Lorentz, *Constructive Approximation*, Springer-Verlag, New York, (1993).

[8] Dolzhenko, E. P., E. A, Sevast'yanov, *Sing-sensitive approximation*, Journal of Mathematical Sciences, Vol. 91, 5, (1998), 3205-3257.

[9] Guerra, F., M. A. Jiménez, itA semi-infinite programming approach to a mixed approximation problem, Parametric Optimization and Related Topics IV, J.Guddat et al, series Approximation and Optimization, Peter Lang Verlag, Frankfurt, (1997), 135-143.

[10] Guerra, F., M. A. Jiménez, *On feasibles sets defined through Chebyshev approximation*, Mathematical Methods of Operations Research, Vol. 47, 2, (1998), 255-264.

[11] Jiménez M. A., E. L. Juárez, F. Guerra, *Transformation of some mixed approximation problems by optimization methods*, Optimization, Vol. 5, 1, (2002), 175-190.

[12] Jiménez M. A., I. L. Martínez, *Mejor aproximación polinomial en bandas no uniformes*, Tópicos de Teoría de Aproximación II, Textos Científicos, Puebla, México, (2007), 97-109.

[13] Jiménez M. A., I. L. Martínez, *Varying amplitude as degree of approximation*, International conference on approximation methods and numerical modeling in environment and natural resources, MAMERN'07, Book of extended abstracts, Granada, España, (2007), 85-88.

[14] Karlin, S., W. J. Studdent, *Tchebycheff Systems: With application in Analysis and Statistics*, Interscience Publishers (1966).

[15] Krein, M. G., A. A. Nudel'man, *The Markov moment problem and extremal problems. Ideas and problems of P. L. Chebyshev and A. A. Markov and their further development*, Nauka, Moscow (1973).

[16] Martínez, I. L., *Aproximación e Interpolación Polinomial en Bandas No Uniformes*, Tesis de doctorado. FCFM-BUAP, (2009).

[17] Mayor, G., O. Valero, *Aggregation of asymmetric distances in Computer Science*, Information Sciences, Volume 180, Issue 6, (2010), 803-812.

[18] Méndez, J. N., *Problema de la Unicidad de Aproximación con Sistemas Haar Mediante Bandas de Amplitudes Variantes*, Tesis de maestría, FCFM-BUAP, (2010).

[19] Méndez, V. M., it Desarrollo Computacional de la Aproximación polinomial algebraica y trigonométrica mediante bandas de amplitudes variantes, Tesis de maestría, FCFM-BUAP, (2010).

[20] Moursund, D. G., *Chebyshev approximation using a generalized weight function*, SIAM Journal on Numerical Analysis, Vol. 3, 3, (1966), 435-450.

[21] Moursund, D. G, *Computational aspects of Chebyshev approximation using a generalized weight function*, SIAM Journal on Numerical Analysis, Vol. 5, (1968), 126-137.

[22] Pokrovskii, A. V., *The best asymmetric approximation in spaces of continuous functions*, Izvestiya Mathematics, Vol. 70, 4, (2006), 809-839.

[23] Remez, E. Ya., *Foundations of numerical methods of Chebyshev approximation*, Naukova Dumka, Kiev 1969 (Russian).

[24] Tapia, R. A.,M. W. Trosset, An extension of the Karush-Kuhn-Tucker: necessity conditions to infinite programming, SIAM Review, Vol. 36, No. 1, 1994, 1-17.

[25] Tikhomirov, V. M., *Commentary.* Journal of Approximation Theory, 106, (2000), 58-65.

8

Mejora del desempeño de una lente triplete con el método de relajación

Lidia A. Hernández Rebollar[1], **César A. García Vázquez**[2], **Francisco J. Renero Carrillo**[3]

Abstract: Para obtener un mejor desempeño en un sistema óptico denominado triplete, comúnmente usado en sistemas formadores de imagen, se usa el algoritmo ALSIE (Automatic Lens design by Solving InEqualities). En este trabajo se presenta una actualización al método de relajación, empleado para la solución de sistemas de desigualdades lineales, y que es parte fundamental de ALSIE. Los resultados, evaluados mediante la función de mérito de residuos, muestran que el desempeño es un 10% mejor con la actualización presentada que con el algoritmo original.

[1]Facultad de Ciencias Físico Matemático, BUAP
[2]Facultad de Ciencias de la Computación, BUAP
[3]INAOE

8.1 Introducción: Diseño Automático de lentes mediante la solución de desigualdades.

Sean f_i las funciones de rendimiento de un sistema de lentes, estas funciones describen aberraciones ópticas, condiciones mecánicas u otras características del sistema. Cada función de rendimiento es una función continua de $x \in \mathbb{R}^n$, donde x es el vector de parámetros del sistema de lentes, como son, radio de curvatura, grosor, etc.. El concepto de diseño automático de lentes se refiere a los métodos para obtener un sistema de lentes óptimo a partir de un sistema de lentes dado. Este problema originalmente parte de la solución del sistema de ecuaciones

$$f_i\left(x\right) = \mu_i \qquad i = 1, 2, ..., m$$

donde μ_i son los valores esperados de las funciones de rendimiento. Puesto que las funciones f_i son no lineales y el sistema es no consistente, no se tiene solución exacta. Por esta razón, los métodos convencionales de diseño automático de lentes se proponen hallar el vector de parámetros x que minimice la función de mérito

$$\Phi = \sum_{i=1}^{m} w_i^2 \left(f_i\left(x\right) - \mu_i\right)^2$$

donde cada w_i es un factor de peso que indica la importancia de la función de rendimiento f_i.

El programa ALSIE reportado en [1] establece un sistema de cotas para los valores esperados de las funciones de rendimiento y transforma el sistema de ecuaciones en el sistema de inecuaciones:

$$\alpha_i \leq f_i\left(x\right) \leq \beta_i, \qquad i = 1, 2, ..., m$$

donde $\mu_i \in [\alpha_i, \beta_i]$. Como las funciones de rendimiento son no lineales, para un x^0 dado, el cual representa el estado actual del sistema de lentes, ALSIE utiliza el gradiente para linealizar y obtiene el siguiente sistema lineal de inecuaciones:

$$\alpha_i \leq \nabla f_i\left(x^0\right)\left(x - x^0\right) + f_i\left(x^0\right) \leq \beta_i, \qquad i = 1, 2, ..., m$$

Definiendo:

$$C = \begin{bmatrix} c_1 & . & . & 0 \\ . & . & & . \\ . & & . & . \\ 0 & . & . & c_n \end{bmatrix}$$

$$y^0 = C^{-1}x^0$$

$$a_i = \begin{cases} \left[\nabla f_i\left(x^0\right)\right]C, & i = 1, 2, ..., m \\ -\left[\nabla f_i\left(x^0\right)\right]C, & i = m+1, ..., 2m \end{cases}$$

$$b_i = \begin{cases} f_i\left(x^0\right) - \left[\nabla f_i\left(x^0\right)\right]Cy^0 - \alpha_i, & i = 1, 2, ..., m \\ -f_i\left(x^0\right) + \left[\nabla f_i\left(x^0\right)\right]Cy^0 - \beta_i, & i = m+1, ..., 2m \end{cases}$$

Se obtiene el sistema:

$$a_i y + b_i \geq 0, \quad i = 1, 2, ..., 2m$$

el cual, ALSIE procede a resolver por el método de relajación propuesto por [2] . Si el punto hallado satisface el sistema no lineal de desigualdades se termina, y si no, se toma este como el nuevo punto inicial x^0, y se realiza otra iteración.

En este trabajo, presentamos el método de relajación descrito por Goberna en [3] , el cual es más simple que el usado en ALSIE por ser este un caso particular, lo hemos programado en Sage, el cual es un software de acceso libre, y hemos obtenido mejores resultados. Para comparar los resultados, se han evaluado los puntos hallados en la función de mérito Φ. El punto hallado con Sage satisface estrictamente el sistema lineal de desigualdades y el no lineal también, además, proporciona un valor menor para Φ que el obtenido con ALSIE.

8.2 El método de relajación

Sea $\{a_i^T x \geq b_i, i \in I\}$ el sistema dado, en el que se puede suponer $a_i \neq 0$, para todo $i \in I$, de modo que cada inecuación representa un semiespacio. Sea F su conjunto solución. La idea geométrica del método es muy simple: supóngase que el punto actual, $\bar{x}$, no es solución del sistema (es decir, $\bar{x} \notin F$); de entre los hiperplanos asociados con inecuaciones violadas por $\bar{x}$ se toma uno de los más alejados de $\bar{x}$, llamándolo H; el punto siguiente se obtiene desplazando $\bar{x}$ perpendicularmente hacia H una distancia igual a $\lambda d(\bar{x}, H)$, donde $\lambda > 0$ es un parámetro prefijado. Como la distancia euclidea desde $\bar{x}$ al hiperplano $H = \{x \in \mathbb{R}^n | a^T = b\}$ tal que $a^T\bar{x} < b$ viene dada por el escalar $\mu = d(\bar{x}, H) = \frac{b - a^T\bar{x}}{\|a\|} > 0$, el punto siguiente será $\bar{x} + \lambda\mu\frac{a}{\|a\|}$.
En particular, si se toma $\lambda = 2$ ($\lambda = 1$), el punto siguiente a $\bar{x}$ es el simétrico de $\bar{x}$ respecto de H (la proyección ortogonal de $\bar{x}$ sobre H, respectivamente).

8.2.1 Algoritmo de Relajación

Fije el parámetro de relajación $\lambda > 0$. Tome el índice de iteración $r = 0$ y $x^0 \in \mathbb{R}^n$ arbitrario.

Etapa 1: Sea $I_r = \{i \in I | a_i^T x^r < b_i\}$ (índices de las restricciones violadas). Si $I_r = \emptyset$, fin (x^r es la solución buscada). Si $I_r \neq \emptyset$, continúe.
Etapa 2: Sea

$$\mu_r = \frac{b_{j_r} - a_{j_r}^T x^r}{\|a_{j_r}\|} = \max\left\{\frac{b_i - a_i^T x^r}{\|a_i\|}, i \in I_r\right\} \tag{8.1}$$

Tome $x^{r+1} = x^r + \lambda\mu_r\frac{a_{j_r}}{\|a_{j_r}\|}$. Sustituya r por $r + 1$ y vuelva a la Etapa 1.[4]

$\square$

[4] Goberna, Jornet y Puente, *Optimización Lineal: Teoría, Métodos y Modelos*, McGraw-Hill, 2004.

8.3 ¿Qué es Sage?

SAGE es un sistema algebraico computacional (en inglés CAS) escrito en Python y en una versión modificada de Pyrex (llamada inicialmente SageX y posteriormente Cython). Reune y unifica bajo un solo entorno, lenguaje y jerarquía de objetos toda una colección de software matemático y trata de rellenar los huecos de funcionalidad dejados por unos y otros. Proporciona un interfaz Python al siguiente software libre: GAP, Pari, Maxima, SINGULAR (todos distribuidos con SAGE). También proporciona un interfaz a software no libre: Magma, Maple, Mathematica (no distribuidos con SAGE). Puede usarse en modo texto o en modo gráfico, accediendo a través de un navegador al servidor web que incluye.[5]

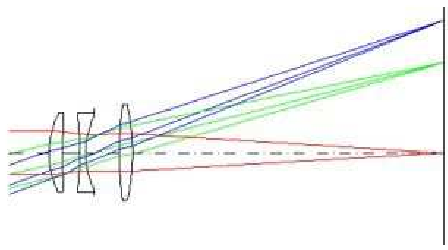


Figure 8.1: Comparación de las soluciones aproximada y exacta

[5] http://es.wikipedia.org/wiki/SAGE

8.4 Codificación del método de relajación en Sage

En el código 8.1 se presenta el método de relajación programado para su utilización en Sage. En general, el método está programado en *Python* salvo las funciones utilizadas por Sage para poder realizar los cálculos de manera más rápida y reducir las líneas de código.

Código 8.1: Método de relajación en Sage

```
def relajacion(aij, bi, LAMBDA):

    #Se comienza obteniendo el n\'umero de variables
    #del problema asi como el numero de inecuaciones
    #(restricciones) del mismo

    inecuaciones = aij.nrows()
    variables = aij.ncols()

    #Para mejorar el desempenio del tiempo de ejecucion
    #del codigo, se asigna desde el comienzo la memoria
    #que se va a utilizar asi como la declaracion de
    #ciertas variables que son importantes dentro de
    #la ejecucion

    #En este caso se declara el arreglo que guardara
    #las normas de cada inecuacion dada

    norma = aij[:,0] - aij[:,0]
    norma = norma.transpose()

    #Xi es la solucion inicial que se utiliza
    #para evaluar la solucion actual

    Xi = zero_matrix(1, variables)
```

```

    #El arreglo Mu se utiliza para guardar el valor de
    #mu de acuerdo al algoritmo, en este caso Mu(1,1)
    #ya guarda el maximo valor de mu, en Mu(1,2) se
    #guarda la evaluacion actual para ser compara mas
    #adelante y finalmente Mu(1,3) guarda el indice del
    #valor maximo de mu (r)

    Mu = aij[0,0:2] - aij[0,0:2]

    #Se da un valor inicial que no afecta al algoritmo
    #pero es necesario si la solucion inicial propuesta
    #es una solucion del sistema

    #En el siguiente ciclo for se obtienen las normas de
    #todas las inecuaciones para evitar realizar el
    #recalculo cada vez que son llamadas

    for indice1 in range(0, inecuaciones):
        for indice2 in range(0,variables):
            num = aij[indice1, indice2]
            num = float(num)
            num = num**2.0
            sum = float(norma[0,indice1])
            sum = sum + num
            norma[0,indice1] = sum
        norma[0,indice1] = sqrt( norma[0,indice1] )

    #A continuacion se determina una solucion inicial
    #del problema de forma aleatoria.

    for indice1 in range(0,variables):
        Xi[0, indice1] = 1

```

```
    #La variable indiceIndices determina el
    #indice que se tiene en el arreglo de los
    #indices que no cumplen con las condiciones

    indiceIndices = 1
    superIndice = 0

    #Aqui comienza el algoritmo, este ciclo se
    #ejecutar\'a mientras el conjunto de indices
    #no se encuentre vacio

    Mu[0,0] = 0.0
    while indiceIndices >= 1:

        #En esta parte se determina la siguiente solucion
        #del sistema

        denNor = norma[0, superIndice]
        vectorAsumar = LAMBDA*Mu[0,0]*aij[ superIndice, :] denNor
        Xi = Xi + vectorAsumar

        #Se comienza asumiendo que todas las restricciones
        #son cumplidas

        indiceIndices = 0

        #Se inicializa Mu(1,1) para evitar comparar el valor
        #maximo de mu de otros ciclos

        Mu[0,0] = 0.0
        for indice1 in range(0,inecuaciones):
            evaluacion = aij[indice1, :]*Xi.transpose()
            evaluacion = evaluacion[0,0]
            biActual = bi[ indice1]
```

```
        if evaluacion <= biActual:
            indiceIndices = indiceIndices + 1
            denNor = norma[ 0,indice1 ]
            Mu[0,1] = ( biActual - evaluacion )/ denNor
            if Mu[0, 1] > Mu[0,0]:
                Mu[0,0] = Mu[0,1]
                superIndice = indice1
    return Xi
```

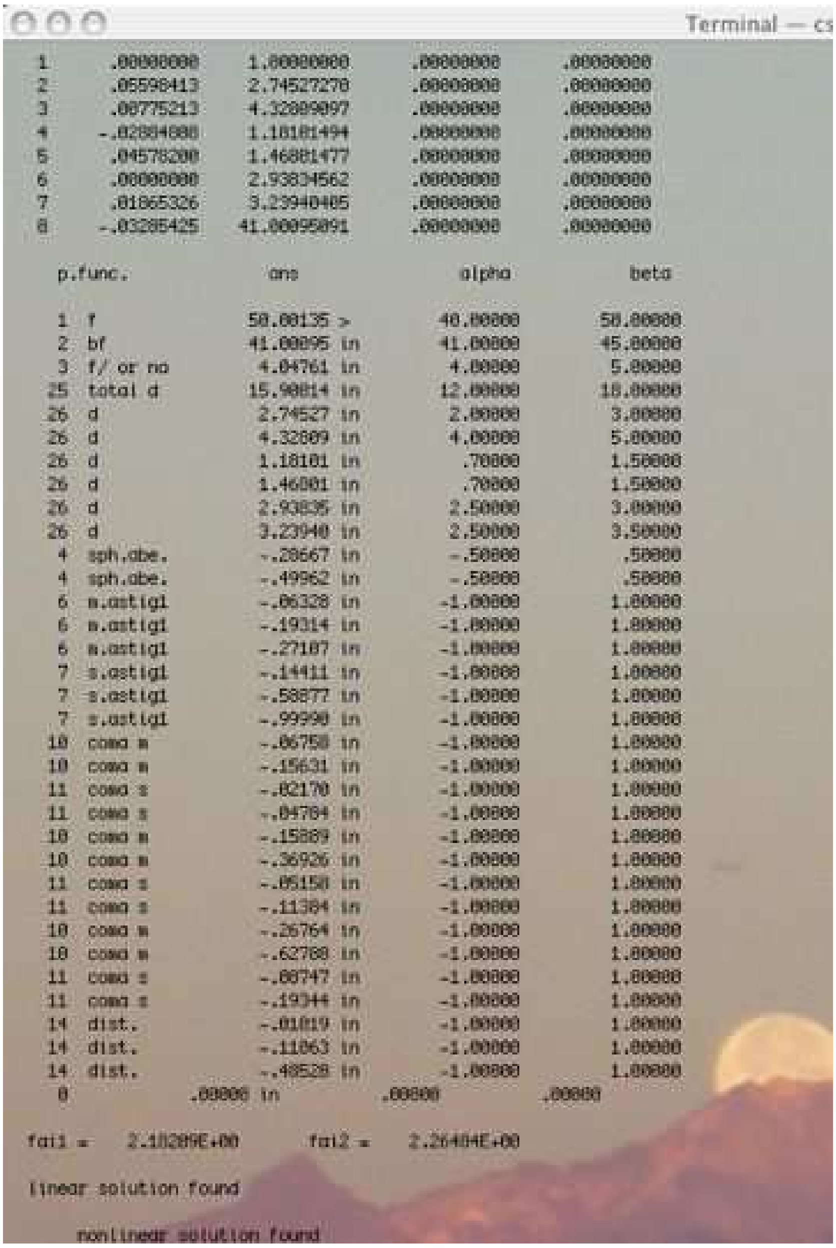
8.5 Resultados

En la figura 8.1 se muestra la esquematización de un triplete clásico para cámara fotográfica. La búsqueda de los parámetros del triplete, es para obtener distancias focales compactas (con respecto al triplete de inicio), de entre $40 - 50$ y $41 - 45$ mm., para la efectiva y posterior, respectivamente.

La Figura 8.2 es una fotografía de la pantalla que muestra los resultados obtenidos con ALSIE para el triplete clásico. En el lado izquierdo se enlistan las 33 funciones de rendimiento, enseguida el valor de estas funciones en el punto solución y en las dos columnas de la derecha los valores de α y β para cada una de las funciones. Abajo de estas columnas fai1$=2.18209$ es el valor de la función de mérito en el punto hallado por ALSIE.

Al ejecutar el método de relajación programado en Sage se obtiene:

```
sage: time relajacion(aij, bi, 1.875)
CPU times: user 66.29 s, sys: 0.72 s, total: 67.02 s
Wall time: 67.14 s
[ 10000000.015     100000.03      9999999.97442
  100000.05        9999999.93589  100000.007203
  10000000.0119    100000.015     100000.03
  9999999.97518    100000.026124  9999999.90935]
```



```
Terminal — cs

 1      .00000000    1.00000000     .00000000     .00000000
 2      .05598413    2.74527270     .00000000     .00000000
 3      .00775213    4.32809097     .00000000     .00000000
 4     -.02884088    1.18181494     .00000000     .00000000
 5      .04578200    1.46881477     .00000000     .00000000
 6      .00000000    2.93834562     .00000000     .00000000
 7      .01865326    3.23940405     .00000000     .00000000
 8     -.03285425   41.00095091     .00000000     .00000000

   p.func.               ans              alpha          beta

   1  f                50.00135 >        40.00000       50.00000
   2  bf               41.00095 in       41.00000       45.00000
   3  f/ or no          4.04761 in        4.00000        5.00000
  25  total d          15.90814 in       12.00000       18.00000
  26  d                 2.74527 in        2.00000        3.00000
  26  d                 4.32809 in        4.00000        5.00000
  26  d                 1.18101 in         .70000        1.50000
  26  d                 1.46001 in         .70000        1.50000
  26  d                 2.93835 in        2.50000        3.00000
  26  d                 3.23940 in        2.50000        3.50000
   4  sph.abe.          -.28667 in        -.50000         .50000
   4  sph.abe.          -.49962 in        -.50000         .50000
   6  m.astigl          -.06328 in       -1.00000        1.00000
   6  m.astigl          -.19314 in       -1.00000        1.00000
   6  m.astigl          -.27107 in       -1.00000        1.00000
   7  s.astigl          -.14411 in       -1.00000        1.00000
   7  s.astigl          -.58877 in       -1.00000        1.00000
   7  s.astigl          -.99990 in       -1.00000        1.00000
  10  coma m            -.06758 in       -1.00000        1.00000
  10  coma m            -.15631 in       -1.00000        1.00000
  11  coma s            -.02170 in       -1.00000        1.00000
  11  coma s            -.04704 in       -1.00000        1.00000
  10  coma m            -.15889 in       -1.00000        1.00000
  10  coma m            -.36926 in       -1.00000        1.00000
  11  coma s            -.05158 in       -1.00000        1.00000
  11  coma s            -.11384 in       -1.00000        1.00000
  10  coma m            -.26764 in       -1.00000        1.00000
  10  coma m            -.62788 in       -1.00000        1.00000
  11  coma s            -.08747 in       -1.00000        1.00000
  11  coma s            -.19344 in       -1.00000        1.00000
  14  dist.             -.01019 in       -1.00000        1.00000
  14  dist.             -.11063 in       -1.00000        1.00000
  14  dist.             -.48528 in       -1.00000        1.00000
   0            .00000 in          .00000         .00000

fai1 =    2.18289E+00        fai2 =    2.26404E+00

linear solution found

     nonlinear solution found
```

Figure 8.2: Comparación de las soluciones aproximada y exacta

También se ejecuta un programa para verificar que esta solución es la correcta. En la tabla 8.2 se aprecia que el resultado obtenido por Sage es válido.

-662916948.960342	>	-662916953.420960
-2238372022.128376	>	-2238372026.128374
-146259206.626580	>	-146259206.626580
60000015.832655	>	60000011.999999
10000003.000000	>	10000002.000000
10000004.999999	>	10000003.999999
10000000.720256	>	10000000.699999
10000001.500000	>	10000000.700000
10000002.999985	>	10000002.500000
10000002.612417	>	10000002.500001
-352816092.370869	>	-352816093.080647
-756456648.948313	>	-756456649.948312
55802004.956334	>	55802004.118924
197558608.888775	>	197558608.214170
152028713.942227	>	152028711.942227
22485505.827176	>	22485505.004545
95029680.045839	>	95029679.729879
178015214.220085	>	178015214.220085
-10905092.382257	>	-10905093.513717
-27362634.672168	>	-27362635.986423
-3454624.313215	>	-3454625.355034
-8096013.417458	>	-8096014.512048
-17132997.702712	>	-17132998.972357
-43563144.184187	>	-43563145.841334
-5467013.551473	>	-5467014.636997
-13262561.338852	>	-13262562.534239
-19647619.398322	>	-19647620.786075
-45769256.932371	>	-45769258.899787
-2132915.825079	>	-2132916.933808
-6817962.971514	>	-6817964.225459

-37101259.632715	>	-37101260.605271
-228044893.484360	>	-228044894.476381
-732142761.248398	>	-732142762.791388
662916948.960342	>	662916943.420960
2238372022.128376	>	2238372022.128374
146259206.626580	>	146259205.626580
-60000015.832655	>	-60000017.999999
-10000003.000000	>	-10000003.000000
-10000004.999999	>	-10000004.999999
-10000000.720256	>	-10000001.499999
-10000001.500000	>	-10000001.500000
-10000002.999985	>	-10000003.000000
-10000002.612417	>	-10000003.500001
352816092.370869	>	352816092.080647
756456648.948313	>	756456648.948312
-55802004.956334	>	-55802006.118924
-197558608.888775	>	-197558610.214170
-152028713.942227	>	-152028713.942227
-22485505.827176	>	-22485507.004545
-95029680.045839	>	-95029681.729879
-178015214.220085	>	-178015216.220085
10905092.382257	>	10905091.513717
27362634.672168	>	27362633.986423
3454624.313215	>	3454623.355034
8096013.417458	>	8096012.512048
17132997.702712	>	17132996.972357
43563144.184187	>	43563143.841334
5467013.551473	>	5467012.636997
13262561.338852	>	13262560.534239
19647619.398322	>	19647618.786075

45769256.932371	>	45769256.899787
2132915.825079	>	2132914.933808

6817962.971514	$>$	6817962.225459
37101259.632715	$>$	37101258.605271
228044893.484360	$>$	228044892.476381
732142761.248398	$>$	732142760.791388

Tabla 8.2: Verificación de la solución

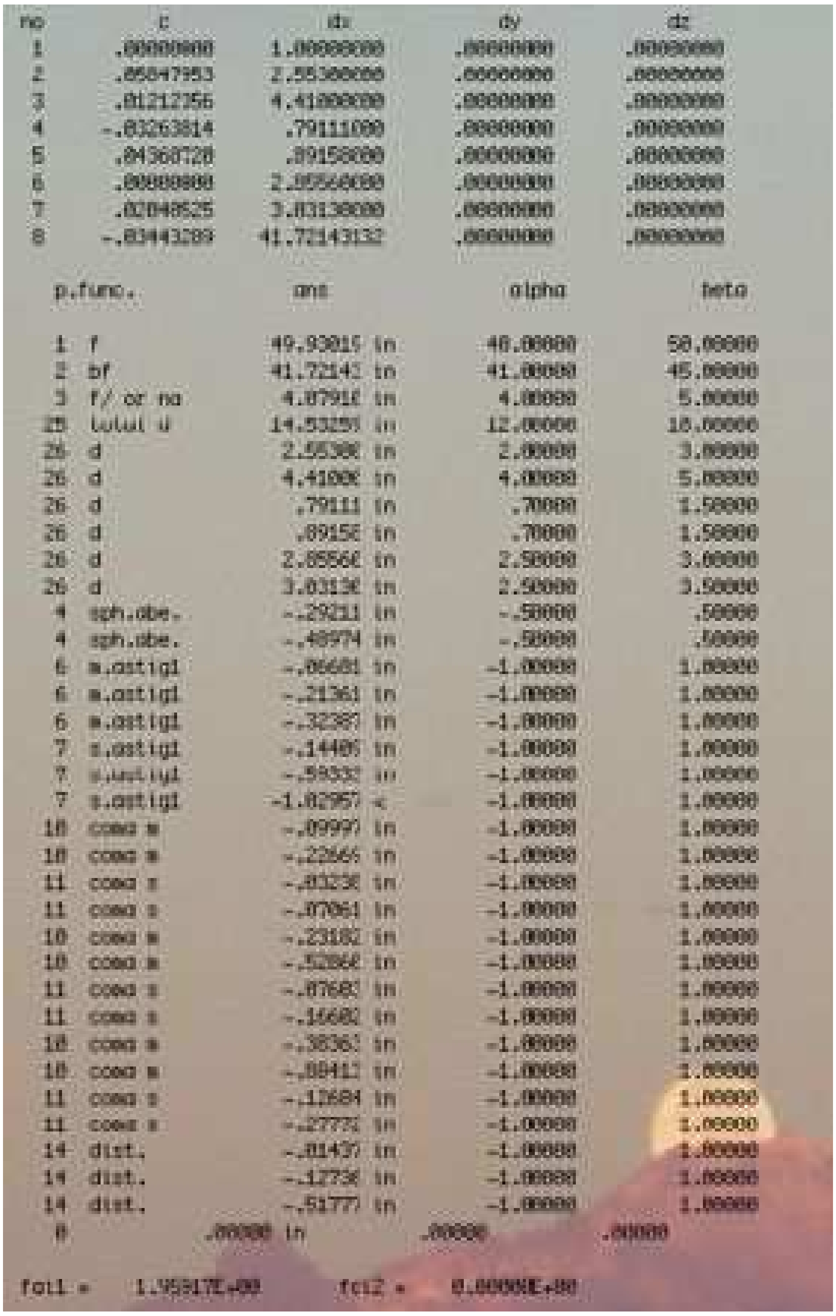
La Figura 8.3 muestra una foto de la pantalla de la evaluación con ALSIE del punto hallado con el programa en Sage. Al final de esta imagen se tiene la evaluación de la función de mérito fai=1.95, la cual es 10% menor que la obtenida con ALSIE.

8.6 Conclusiones

Utilizando el método de relajación presentado en [3] se han hallado mejores parámetros para el sistema de lentes triplete usado en una cámara fotográfica. Afirmamos que estos parámetros son mejores porque proporcionan un valor 10% menor para la función de mérito. Esto se ha logrado porque el método de relajación nos ha proporcionado un punto en el interior del conjunto solución del sistema, y porque el método se ha programado en Sage, un software libre que ha resultado muy eficiente. Por ahora, se ha trabajado por separado con dos algoritmos, uno es ALSIE, el cual lleva a cabo los cálculos para el diseño automático de lentes, y el otro que se ha elaborado en Sage para resolver únicamente el sistema lineal de desigualdades. De ahora en adelante se trabajará en la incorporación del nuevo algoritmo de relajación en ALSIE y se realizarán las pruebas.

8.7 Referencias

[1] Francisco J. Renero Carrillo, Automatic Design of Lens Arrays for Optical Computing and Interconnects, Tesis Doctoral, Osaka University, Japan, 1995.



no	c	dx	dy	dz
1	.00000000	1.00000000	.00000000	.00000000
2	.05847953	2.55300000	.00000000	.00000000
3	.01212356	4.41000000	.00000000	.00000000
4	-.03263814	.79111000	.00000000	.00000000
5	.04360720	.89158000	.00000000	.00000000
6	.00000000	2.85560000	.00000000	.00000000
7	.02040525	3.03130000	.00000000	.00000000
8	-.03443289	41.72143132	.00000000	.00000000

p.func.		ans		alpha	beta
1	f	49.93015	in	48.00000	50.00000
2	bf	41.72143	in	41.00000	45.00000
3	f/ or na	4.07916	in	4.00000	5.00000
25	total d	14.53259	in	12.00000	18.00000
26	d	2.55300	in	2.00000	3.00000
26	d	4.41000	in	4.00000	5.00000
26	d	.79111	in	.70000	1.50000
26	d	.89158	in	.70000	1.50000
26	d	2.85560	in	2.50000	3.00000
26	d	3.03130	in	2.50000	3.50000
4	sph.abe.	-.29211	in	-.50000	.50000
4	sph.abe.	-.40974	in	-.50000	.50000
6	m.astig1	-.06681	in	-1.00000	1.00000
6	m.astig1	-.21361	in	-1.00000	1.00000
6	m.astig1	-.32387	in	-1.00000	1.00000
7	s.astig1	-.14406	in	-1.00000	1.00000
7	s.astig1	-.59332	in	-1.00000	1.00000
7	s.astig1	-1.02957	<	-1.00000	1.00000
10	coma m	-.09997	in	-1.00000	1.00000
10	coma m	-.22669	in	-1.00000	1.00000
11	coma s	-.03238	in	-1.00000	1.00000
11	coma s	-.07061	in	-1.00000	1.00000
10	coma m	-.23182	in	-1.00000	1.00000
10	coma m	-.52866	in	-1.00000	1.00000
11	coma s	-.07683	in	-1.00000	1.00000
11	coma s	-.16682	in	-1.00000	1.00000
10	coma m	-.38363	in	-1.00000	1.00000
10	coma m	-.88413	in	-1.00000	1.00000
11	coma s	-.12684	in	-1.00000	1.00000
11	coma s	-.27773	in	-1.00000	1.00000
14	dist.	-.01437	in	-1.00000	1.00000
14	dist.	-.12738	in	-1.00000	1.00000
14	dist.	-.51777	in	-1.00000	1.00000
0		.00000	in	.00000	.00000

fci1 = 1.95917E+00 fci2 = 0.00000E+00

Figure 8.3: Evaluación del punto hallado con Sage

[2] Yu I. Merzlyakov, On a Relaxation Method of Solving Systems of Linear Inequalities, Zh. Vych. Mat. 2, No. 3, 482-487, 1962

[3] M.A. Goberna, V. Jornet, R. Puente, Optimización Lineal. Teoría Métodos y Modelos, Mc Graw Hill, 2004

[4] Schrijver, A., Theory of Linear and Integer Programming, J. Wiley, Chichester, 1986

9

Stability of Closed-Convex-Valued Mappings: Survey

Maxim Ivanov Todorov [1]

[1]UDLAP, Puebla, MX and IMI-BAS, Sofia, BG, maxim.todorov@udlap.mx

Abstract: In general, the stability in optimization, approximation, etc., is closely related to the continuity properties (closedness, lower and upper semicontinuity, metric regularity, etc.) of several multivalued mappings, like the feasible and optimal set mappings, the mappings which put into correspondence to each parameter (system or problem) the set of the boundary, relative boundary or the extreme points, respectively. Because of this, we have presented some results devoted to the stability of general closed-convex-valued mappings. The first part considers how the continuity properties of the original general closed-convex-valued mapping have been transmitted to the associated boundary mapping, which is not convex valued, and vice versa. The second chapter presents similar questions concerning the boundary, relative boundary or the extreme points set mappings, respectively. The mappings do not always inherit the continuity properties from each others, sometimes additional assumptions are required. We would like to mention the continuity properties, established in the first part, of the truncated mappings which are of a special use in many areas not only in optimization.

Keywords: stability, multivalued mappings

Classification: Primary 49K40; Secondary 28B20.

9.1 Introduction

The main objective of this article is to analyze the relationship between important pairs of mappings, one of them being the convex hull of the other, which frequently arise in convex optimization (convex systems), where, as a consequence of measurement or roundoff errors, the nominal problem y_0 (system y_0) is usually replaced in practice by perturbed problems (systems, respectively) having the same structure. Let us denote by Y the set of all possible perturbed problems (systems) equipped with a certain pseudometric measuring the size of the perturbations and let $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$ be the set-valued mapping associating with each $y \in Y$ its feasible set or its optimal set (its solution set, respectively). Under mild conditions, $\mathcal{F}(y)$ is the convex hull of its boundary set $\mathrm{bd}\mathcal{F}(y)$, its relative boundary set $\mathrm{rbd}\mathcal{F}(y)$ and/or its extreme points set $\mathrm{ext}\mathcal{F}(y)$, for all $y \in Y$. We denote these mappings from Y to $\mathbb{R}^n$ as $\mathrm{bd}\mathcal{F}$, $\mathrm{rbd}\mathcal{F}$ and $\mathrm{ext}\mathcal{F}$, which are called *boundary mapping*, *relative boundary mapping* and *extreme points set mapping* of $\mathcal{F}$, respectively. The connections between the stability properties of $\mathcal{F}$, $\mathrm{bd}\mathcal{F}$ and $\mathrm{ext}\mathcal{F}$ have been already analyzed in the particular context of linear semi-infinite systems ([3] and [4], respectively), where Y is equipped with the pseudometric of the uniform convergence.

More generally, we consider given a convex-valued mapping $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$, where the domain Y is a locally metrizable space (i.e., Y is equipped with the topology induced by an extended distance on Y, δ, taking values on $\mathbb{R}_+ \cup \{+\infty\}$), and its boundary mapping, relative boundary mapping and extreme points set mapping, $\mathrm{bd}\mathcal{F}$, $\mathrm{rbd}\mathcal{F}$ and $\mathrm{ext}\mathcal{F}$. The relationships between $\mathcal{F}$ and $\mathrm{bd}\mathcal{F}$, assuming that

$$\mathcal{F}(y) = \mathrm{conv}\,\mathrm{bd}\mathcal{F}(y), \text{ for all } y \in Y; \tag{9.1}$$

have been studied in ([7]).

We denote by $\mathcal{B} : Y \rightrightarrows \mathbb{R}^n$ the associated boundary mapping of $\mathcal{F}$; i.e.,

$$\mathcal{B}(y) := \mathrm{bd}\mathcal{F}(y), \text{ for all } y \in Y.$$

(If $\mathcal{F}$ is a single-valued mapping from Y to $\mathbb{R}^n$, then $\mathcal{F} \equiv \mathcal{B}$). So (9.1) can be reformulated as $\mathcal{F}(y) = \mathrm{conv}\mathcal{B}(y)$ for all $y \in Y$. Obviously, $\mathcal{B}$ is also a closed-valued mapping.

In the same vein, this part considers the relationships between the stability properties of $\mathcal{F}$, $\mathrm{rbd}\mathcal{F}$ and $\mathrm{ext}\mathcal{F}$, studied in ([8]), assuming that $\mathcal{F} = \mathrm{conv}\,\mathrm{rbd}\mathcal{F}$ and $\mathcal{F} = \mathrm{conv}\,\mathrm{ext}\mathcal{F}$, respectively. The finite dimension of the image space plays a crucial role in those arguments based on the compactness of the unit sphere or on Carathéodory's Theorem.

Some of these relationships are direct consequences of basic results about arbitrary mappings $\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$ and their corresponding *convex hull mappings*, $\mathrm{conv}\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$, which associates to each $y \in Y$ the convex hull of

$\mathcal{A}(y)$, i.e., $(\text{conv}\mathcal{A})(y) = \text{conv}\mathcal{A}(y)$ for all $y \in Y$. Although some results on the transmission of stability properties between $\mathcal{A}$ and conv$\mathcal{A}$ are already known (see, e.g., [11] and [1]), we provide proofs of other results which will be used in the sequel. Thus, for each stability property, we start analyzing the relationships between $\mathcal{A}$ and conv$\mathcal{A}$, and then we exploit the properties of the images of $\mathcal{F}$, rbd$\mathcal{F}$ and ext$\mathcal{F}$ in order to obtain the relationships between these mappings. We deal with the lsc property and also with the usc property and closedness.

Let us introduce some additional notation. Given a nonempty set X of a certain Euclidean space, by aff X, span X, conv X, cone X, and dim X we denote the affine hull, the linear hull, the convex hull, the convex conical hull, and the dimension of aff X, respectively. Moreover, we define cone $\varnothing = \{0_n\}$. The positive polar of a given convex cone X is denoted by X^0 and its lineality space by lin$X$. We denote by $X^\perp$ the orthogonal subspace to a given linear subspace X. From the topological side, rint$X$, int$X$, cl$X$, bd X and rbd$X$ represent the relative interior, the interior, the closure, the boundary and the relative boundary of X, respectively. The Euclidean norm in $\mathbb{R}^n$ will be denoted by $\|.\|$ and the open ball centered at x and radius $\varepsilon > 0$ by $B(x;\varepsilon)$. The null-vector, the open unit ball, and the canonical basis in $\mathbb{R}^n$ will be denoted by 0_n, B_n or simply B, and $\{e_1, \ldots, e_n\}$, respectively. The standard simplex in $\mathbb{R}^{n+1}$ is

$$S := \left\{ (\lambda_1, ..., \lambda_{n+1}) \in \mathbb{R}_+^{n+1} \mid \sum_{i=1}^{n+1} \lambda_i = 1 \right\}.$$

For the sake of completeness, we recall the stability concepts and some basic results for set-valued mappings that we shall consider in this paper. Let $\mathcal{M} : Y \rightrightarrows \mathbb{R}^n$ be a set-valued mapping. Its domain is dom$\mathcal{M} := \{y \in Y \mid \mathcal{M}(y) \neq \emptyset\}$. The following semicontinuity concepts are due to Bouligand and Kuratowski (see [1], Section 1.4, [2]).

We say that $\mathcal{M}$ is *lower semicontinuous* at $y_0 \in Y$ *in the Berge sense* (lsc, in brief) if, for each open set $W \subset \mathbb{R}^n$ such that $W \cap \mathcal{M}(y_0) \neq \emptyset$, there exists an open set $V \subset Y$, containing y_0, such that $W \cap \mathcal{M}(y) \neq \emptyset$ for each $y \in V$. Obviously, $\mathcal{M}$ is lsc at $y_0 \notin$ dom$\mathcal{M}$ and $y_0 \in$ intdom$\mathcal{M}$ if $\mathcal{M}$ is lsc at

$y_0 \in \mathrm{dom}\mathcal{M}$.

$\mathcal{M}$ is *upper semicontinuous* at $y_0 \in Y$ *in the Berge sense* (usc, in brief) if, for each open set $W \subset \mathbb{R}^n$ such that $\mathcal{M}(y_0) \subset W$, there exists an open set $V \subset Y$, containing y_0, such that $\mathcal{M}(y) \subset W$ for each $y \in V$. If $\mathcal{M}$ is usc at $y_0 \notin \mathrm{dom}\mathcal{M}$, then $y_0 \in \mathrm{int}(Y \diagdown \mathrm{dom}\mathcal{M})$.

If $\mathcal{M}$ is simultaneously lsc and usc at y_0 we say that $\mathcal{M}$ is *continuous* at this point.

$\mathcal{M}$ is *closed* at $y_0 \in \mathrm{dom}\mathcal{M}$ if for all sequences $\{y_r\}_{r=1}^{\infty} \subset Y$ and $\{x_r\}_{r=1}^{\infty} \subset \mathbb{R}^n$ satisfying $x_r \in \mathcal{M}(y_r)$ for all $r \in \mathbb{N}$, $\lim_{r\to\infty} y_r = y_0$ and $\lim_{r\to\infty} x_r = x_0$ (in brief, $y_r \to y_0$ and $x_r \to x_0$) one has $x_0 \in \mathcal{M}(y_0)$. If $\mathcal{M}$ is usc at $y_0 \in \mathrm{dom}\mathcal{M}$ and $\mathcal{M}(y_0)$ is closed, then $\mathcal{M}$ is closed at y_0. Conversely, if $\mathcal{M}$ is closed and *locally bounded* at $y_0 \in \mathrm{dom}\mathcal{M}$ (i.e., if there are a neighborhood of y_0, say V, and a bounded set $A \subset \mathbb{R}^n$ containing $\mathcal{M}(y)$ for every $y \in V$), then $\mathcal{M}$ is usc at y_0.

Finally, $\mathcal{M}$ is lsc (usc, closed, locally bounded) if it is lsc (usc, closed, locally bounded) at y for all $y \in Y$.

9.2 Closed-convex-valued mappings and the associated boundaries

This chapter is organized as follows. Section 1 contains the auxiliary concepts and results. Section 2 presents some selected examples, including the case in which Y is formed by the linear inequality systems obtained by perturbing arbitrarily the coefficients of a given (nominal) system. In this particular context, it is known that the solution set mapping $\mathcal{F}$ is closed everywhere and inherits the lsc property from $\mathcal{B}$ (see [3], where it was conjectured that the last statement could be true replacing lsc with usc). Sections 3 and 6 show that $\mathcal{F}$ always inherits the lsc property and the continuity from $\mathcal{B}$, respectively, whereas Section 4 and 5 show similar results for the closedness and the usc property, respectively, under additional assumptions. The lsc property and the continuity of $\mathcal{F}$ also entail the corresponding properties of $\mathcal{B}$ under suitable conditions.

9.2.1 Preliminaries

Following [12] we may have defined $\mathcal{M}$ as being *continuous* at y_0 *in the Bouligand sense* if

$$\liminf_{y\to y_0} \mathcal{M}(y) = \limsup_{y\to y_0} \mathcal{M}(y) = \mathcal{M}(y_0), \tag{9.2}$$

where the sets $\liminf_{y\to y_0} \mathcal{M}(y)$ and $\limsup_{y\to y_0} \mathcal{M}(y)$ are the so-called *inner limit* and *outer limit* respectively. These sets are defined as follows:

$$\begin{aligned}\liminf_{y\to y_0} \mathcal{M}(y) &= \{x \mid \forall y_r \to y_0\ \exists x_r \to x \text{ with } x_r \in \mathcal{M}(y_r)\}\\ &= \bigcap_{y_r\to y_0} \liminf_{r\to\infty} \mathcal{M}(y_r),\end{aligned}$$

and

$$\limsup_{y\to y_0} \mathcal{M}(y) = \{x \mid \exists y_r \to y_0\ \exists x_r \to x \text{ with } x_r \in \mathcal{M}(y_r)\}$$

$$= \bigcup_{y_r\to y_0} \limsup_{r\to\infty} \mathcal{M}(y_r).$$

Obviously $\liminf_{y\to y_0} \mathcal{M}(y) \subset \limsup_{y\to y_0} \mathcal{M}(y)$.

When $\limsup_{y\to y_0} \mathcal{M}(y) = \mathcal{M}(y_0)$, it is said that $\mathcal{M}$ is *outer semicontinuous* at y_0 and, similarly, $\mathcal{M}$ is *inner semicontinuous* (isc) at y_0 if $\liminf_{y\to y_0} \mathcal{M}(y) = \mathcal{M}(y_0)$. Thus, the continuity in the Bouligand sense is equivalent to simultaneous inner and outer semicontinuity, and also to the fact that $\mathcal{M}(y_r)$ *converges in the sense of Painlevè-Kuratowski* to $\mathcal{M}(y_0)$ for all possible sequence $\{y_r\}_{r=1}^{\infty}$ converging to y_0.

In [12] it is stated that, for closed-valued mappings, inner semicontinuity at y_0 is equivalent to lower semicontinuity at y_0, whereas outer semicontinuity at y_0 is equivalent to closedness at y_0.

If $C, D \subset \mathbb{R}^n$ are nonempty compact sets, the *Hausdorff distance* between C and D is

$$d_H(C, D) := \max\{\max_{x\in C} d(x, D), \max_{y\in D} d(y, C)\},$$

where $d(x, A) := \inf_{z \in A} \|x - z\|$. The Hausdorff distance provides a metric in the space of the nonempty compact sets in the Euclidean space $\mathbb{R}^n$, and it can easily be verified that

$$d_H(C, D) = d_H(\mathrm{bd}C, \mathrm{bd}D). \tag{9.3}$$

A sequence $\{C_r\}_{r=1}^{\infty}$ is said to *converge with respect to the Hausdorff distance* to C when $d_H(C_r, C) \to 0$ (all these sets being nonempty and compact in $\mathbb{R}^n$). This property entails the convergence of C_r to C in the sense of Painlevè-Kuratoski, and it is equivalent to it if there exist a bounded set $A \subset \mathbb{R}^n$ containing all C_r, $r = 1, 2, ...,$ and C.

The following result is quite standard in multivalued functions analysis:

Lemma 9.2.1 *Consider a closed-valued mapping $\mathcal{M} : Y \rightrightarrows \mathbb{R}^n$ and a point $y_0 \in \mathrm{dom}\mathcal{M}$. If $\mathcal{M}$ is continuous at y_0 in the sense of Berge then $\mathcal{M}$ will be also continuous at y_0 in the sense of Bouligand, and the converse holds if $\mathcal{M}$ is locally bounded at y_0.*
Moreover, under the last assumption and assuming $y_0 \in \mathrm{intdom}\mathcal{M}$, any continuity at y_0 is equivalent to the following property: for every sequence $\{y_k\}_{k=1}^{\infty}$ converging to y_0, there exists k_0 such that $\mathcal{M}(y_k)$ is a nonempty bounded set for all $k \geq k_0$ and $\{\mathcal{M}(y_k)\}_{k=k_0}^{\infty}$ converges with respect to the Hausdorff distance to $\mathcal{M}(y_0)$.

Given $\mathcal{M} : Y \rightrightarrows \mathbb{R}^n$ and $\rho > 0$, the *truncated mapping* of $\mathcal{M}$ is $\mathcal{M}_\rho : Y \rightrightarrows \mathbb{R}^n$ such that

$$\mathcal{M}_\rho(y) := \mathcal{M}(y) \cap \mathrm{cl}B(0_n; \rho).$$

The next result summarizes the relationships between both mappings in the stability context.

Lemma 9.2.2 *Let $\mathcal{M} : Y \rightrightarrows \mathbb{R}^n$ and let $y_0 \in \mathrm{dom}\mathcal{M}$. Then the following statements hold:*
(i) $\mathcal{M}$ is closed at y_0 if and only if $\mathcal{M}_\rho$ is closed at y_0 for all $\rho > 0$ such that $\mathcal{M}_\rho(y_0) \neq \emptyset$.
(ii) If $\mathcal{M}$ is usc at y_0 and $\mathcal{M}(y_0)$ is closed, then $\mathcal{M}_\rho$ is usc at y_0 for all $\rho > 0$

such that $\mathcal{M}_\rho\,(y_0) \neq \emptyset$.
(iii) If $\mathcal{M}$ *is usc at* y_0*, then there exist a positive scalar* $\overline{\rho}$ *and an open neighborhood of* y_0, V*, such that*

$$\mathcal{M}(y)\diagdown\mathcal{M}_{\overline{\rho}}(y) \subset \mathcal{M}(y_0)\diagdown\mathcal{M}_{\overline{\rho}}(y_0),\ \textit{for all } y \in V. \tag{9.4}$$

The converse statement holds when $\mathcal{M}$ *is closed at* y_0.
(iv) If $\mathcal{M}_\rho$ *is lsc at* y_0 *for every* ρ *such that* $\mathcal{M}(y_0) \cap B\,(0_n;\rho) \neq \emptyset$*, then* $\mathcal{M}$ *is lsc at* y_0*. The converse statement holds if* $\mathcal{M}(y_0)$ *is convex.*

Remark 9.2.1 Among proof of Part (iii), we have given, there exists another proof based upon the so-called Dolecki condition (see, for instance, [2] Lemma 2.2.2) can be found in Theorem 3.1 of [6]. Concerning the direct statement of (iv), it can also be derived from a result about intersection mappings in [9].

Example 9.2.1 Consider the mapping $\mathcal{F} : Y \rightrightarrows \mathbb{R}^2$, where $Y = [0,1]$, $\mathcal{F}(y) = \operatorname{conv}\mathcal{B}(y)$, and

$$\mathcal{B}(y) := \{\lambda(-y,1) \mid \lambda \geq 0\} \cup \{\lambda(1,-y) \mid \lambda \geq 0\}.$$

Obviously $\mathcal{F}_\rho$ is usc at 0 for all $\rho > 0$, but $\mathcal{F}$ is not usc at 0, so that the converse statement of (ii) is not valid in general, even for mappings satisfying (9.1).

The next example shows that the assumption $F\,(Y) \subset R^n$ is essential, so that the results in this part involving the usc property could fail for infinite dimensional spaces.

Example 9.2.2 Let $\Lambda = [0,1]$ and X be the space of finitely nonzero sequences (i.e., $X = \{x = (\zeta_1, \zeta_2, ..., \zeta_i, ...) \mid \zeta_i \in \mathbb{R}, i = 1,2,...,$ and only a finite number of ζ_i are nonzero$\}$), with the supremum norm (i.e., $\|x\| = \max|\zeta_i|$), The mapping $F : \Lambda \rightsquigarrow X$ such that $F\,(\lambda) := \{x \in X \mid \|x\| = \lambda\}$ satisfies trivially our condition in Lemma 9.2.2(iii) but fails to be usc at $\lambda = 1$ (if we take $\lambda_k = (k-1)\diagup k$, $k = 1,2,...$, it evident that the element $u_k \in X$, which has, as the unique nonzero component $\zeta_k = (k-1)\diagup k$, satisfies $u_k \in F\,(\zeta_k)\diagdown F\,(1)$, but the sequence $\{u_k\}_{k=1}^{\infty}$ has no accumulation point and so, the standard Dolecki's condition [2], [Lemma 2.2.2] fails).

9.2.2 Some exploratory examples

The next examples lead us to infer the conjectures on the stability of $\mathcal{F}$ and $\mathcal{B}$ which are checked in the following sections. The academic Examples 2-4 are intended to separate the stability properties of $\mathcal{F}$ and $\mathcal{B}$, which are globally (and so locally) bounded everywhere. Examples 5-6 deal with a mapping connected with the Gauss-Lucas Theorem relating the zeros of a complex polynomial and those of its derivative: the convex hull of all zeros. In Example 7, $\mathcal{F}$ represents the solution set of a linear inequality system subject to arbitrary perturbations of the coefficients (to the authors knowledge this is the unique set-valued mapping whose stability properties have been compared with the stability properties of its boundary mapping). It can be realized that all the mappings represented with $\mathcal{F}$ in Examples 2-7 satisfy condition (9.1). Finally, Example 8 shows that this condition is necessary in order to guarantee the transfer of stability properties from $\mathcal{B}$ to $\mathcal{F}$.

Example 9.2.3 Consider the mapping $\mathcal{F} : \mathbb{R} \rightrightarrows \mathbb{R}$ such that

$$\mathcal{F}(y) := \begin{cases} [-2, 2], & \text{if } y \in \mathbb{Q}, \\ [-1, 1], & \text{otherwise,} \end{cases}$$

where $\mathbb{Q}$ represents the set of rational numbers in $\mathbb{R}$. Here $\mathcal{F}$ is usc (closed) at y_0 if and only if $y_0 \in \mathbb{Q}$, and $\mathcal{F}$ is lsc at y_0 if and only if $y_0 \in \mathbb{R} \diagdown \mathbb{Q}$ (so that it is nowhere continuous in the sense of Berge). Nevertheless, $\mathcal{B}$ is unstable everywhere in all senses.

Example 9.2.4 Consider the mapping $\mathcal{F} : \mathbb{R} \rightrightarrows \mathbb{R}$ such that

$$\mathcal{F}(y) := \begin{cases} [-\left|y\right|^{-1}, \left|y\right|^{-1}], & \text{if } y \neq 0, \\ [-1, 1], & \text{if } y = 0, \end{cases}$$

so that $\mathcal{F}(0)$ is a convex body. Concerning the three basic properties (lsc, usc, closedness) at 0, it is easy to see that $\mathcal{B}$ is only closed and $\mathcal{F}$ is only lsc at that point.

Example 9.2.5 Consider the mapping $\mathcal{F} : \mathbb{R} \rightrightarrows \mathbb{R}$ such that

$$\mathcal{F}(y) := \begin{cases} [0, \sin^2 |y|^{-1}], & \text{if } y \neq 0, \\ \{0\}, & \text{if } y = 0. \end{cases}$$

It can be observed that $\mathcal{F}$ and $\mathcal{B}$ are lsc but neither closed nor usc at 0.

Example 9.2.6 Let Y be the set of nonzero polynomials with real coefficients and degree at most 2. Y can be identified with $\mathbb{R}^3 \setminus \{0_3\}$, $\mathbb{R}^3$ equipped with any norm. Given $y \in Y$, we denote by $\mathcal{Z}(y)$ its set of real zeros and by $\mathcal{F}(y)$ its convex hull, i.e., $\mathcal{F}(y) = \text{conv}\mathcal{Z}(y)$. In this case $\mathcal{B}(y) = \mathcal{Z}(y)$.
$\mathcal{B}$ is closed everywhere: Assume that $x_r \in \mathcal{B}(y_r)$, $r = 1, 2, \ldots$, $x_r \to x_0$ and $y_r \to y_0$. Since $y_r(x_r) = 0$ for all r, taking limits as $r \to +\infty$ we get $y_0(x_0) = 0$, i.e., $x_0 \in \mathcal{Z}(y_0) = \mathcal{B}(y_0)$.
$\mathcal{F}$ and $\mathcal{B}$ are usc (but not lsc) at $y_0 := x^2$ (consider $y_r = x^2 + r^{-1}$, $r = 1, 2, \ldots$). So $\mathcal{F}$ is also closed at y_0.
Now, let $y_0 := -x + 1$. It is easy to see that $\mathcal{F}$ and $\mathcal{B}$ are lsc at y_0. Nevertheless, taking $y_r = r^{-1}x^2 - x + 1$, $r = 1, 2, \ldots$, since $\mathcal{B}(y_r) = \left\{\frac{r \pm \sqrt{r^2 - 4r}}{2}\right\}$, with $\lim_r \frac{r + \sqrt{r^2 - 4r}}{2} = +\infty$ and $\lim_r \frac{r - \sqrt{r^2 - 4r}}{2} = 1$, none of the mappings is usc at y_0 and $\mathcal{F}$ is not closed at y_0 (take $x_r = 2$, $r = 1, 2, \ldots$). So $\mathcal{B}$ is continuous in the sense of Bouligand at y_0 but $\mathcal{F}$ does not.

Example 9.2.7 Let Y be the set of polynomials of degree $q \in \mathbb{N}$ (fixed) with complex coefficients. Since the field of complex numbers can be identified with $\mathbb{R}^2$, Y can be identified with $\mathbb{R}^{2q} \times (\mathbb{R}^2 \setminus \{0_2\})$, $\mathbb{R}^{2q+2}$ equipped with the Euclidean norm. Given $y \in Y$, we denote by $\mathcal{Z}(y)$ its set of complex zeros and by $\mathcal{F}(y)$ its convex hull, i.e., $\mathcal{F}(y) = \text{conv}\mathcal{Z}(y)$ (a polytope in $\mathbb{R}^2$). By the fundamental theorem of algebra, $\mathcal{Z}(y) \neq \emptyset$ for all $y \in Y$, so that $\text{dom}\mathcal{F} = Y$. We shall prove that $\mathcal{F}$ is stable everywhere in all senses (we shall prove in Section 7 that the same is true for $\mathcal{B}$). To do this we shall use the following well-known consequence of Rolle's Theorem for complex polynomials (see, e.g., [10]): Let $y \in Y$, let $\mathcal{Z}(y) = \{w_1, \ldots, w_k\}$, and let n_j be the order of w_j, $j = 1, \ldots, k$. Let

$$0 < \eta < \varepsilon := \frac{1}{2} \min \{|w_j - w_i|, 1 \leq i < j \leq k\}.$$

Then there exists $\delta > 0$ such that $y' \in Y$ has exactly n_j zeros (counted with multiplicity) in $B(w_j; \eta)$, $j = 1, ..., k$, provided that $d(y', y) \leq \delta$.
$\mathcal{F}$ is lsc everywhere: Let $y \in Y$ as above and let W be an open set such that $W \cap \mathcal{F}(y) \neq \emptyset$. Let $z \in W \cap \mathcal{F}(y) = W \cap \operatorname{conv} Z(y)$. Let η be such that $0 < \eta < \varepsilon$ and $\operatorname{cl} B(z; \eta) \subset W$, and let $z = \sum_{j=1}^{k} \lambda_j w_j$, with $\sum_{j=1}^{k} \lambda_j = 1$ and $\lambda_j \geq 0$, $j = 1, ..., k$. Let $y' \in Y$ such that $d(y', y) \leq \delta$ and let $w'_1, ..., w'_q$ be the zeros of y' (possibly repeated). Consider the following partition of $\{1, ..., q\}$:

$$I_j := \{i \in \{1, ..., q\} \mid w'_i \in B(w_j; \eta)\}, j = 1, ..., k,$$

with $|I_j| = n_j$ according to the mentioned result. Let $z' := \sum_{j=1}^{k} \sum_{i \in I_j} \frac{\lambda_j}{n_j} w'_i$. Since $\sum_{j=1}^{k} \sum_{i \in I_j} \frac{\lambda_j}{n_j} = 1$, $z' \in \operatorname{conv}\left\{w'_1, ..., w'_q\right\} = \mathcal{F}(y')$. On the other hand,

$$\begin{aligned} |z - z'| &= \left| \sum_{j=1}^{k} \lambda_j \left(w_j - \sum_{i \in I_j} \frac{1}{n_j} w'_i \right) \right| \\ &\leq \sum_{j=1}^{k} \lambda_j \left| \sum_{i \in I_j} \frac{1}{n_j} (w_j - w'_i) \right| \leq \sum_{j=1}^{k} \lambda_j \sum_{i \in I_j} \frac{\eta}{n_j} = \eta. \end{aligned}$$

Hence $z' \in \operatorname{cl} B(z; \eta) \subset W$ and, so, $W \cap \mathcal{F}(y') \neq \emptyset$.
$\mathcal{F}$ is usc everywhere: Let $y \in Y$ as above and let W be an open set such that $\mathcal{F}(y) \subset W$. Let η be such that $0 < \eta < \varepsilon$ and $\mathcal{F}(y) + B(0_2; \eta) \subset W$. Then, if $y' \in Y$ satisfies $d(y', y) \leq \delta$, maintaining the notation of the previous paragraph, we have $w'_i \in B(w_j; \eta)$ for all $i \in I_j$, $j = 1, ..., k$. Thus

$$\mathcal{F}(y') \subset \operatorname{conv}\left[\bigcup_{j=1}^{k} B(w_j; \eta)\right] = \mathcal{F}(y) + B(0_2; \eta) \subset W.$$

Now, we can assert that $\mathcal{F}$ is continuous in the sense of Berge (and Bouligand) everywhere.

Example 9.2.8 Let $n \geq 2$ and let T be an arbitrary set such that $|T| \geq 2$. We associate with $y = (a, b)$, where $a : T \to \mathbb{R}^n$ and $b : T \to \mathbb{R}$, the linear system $\{a_t' x \geq b_t, t \in T\}$, in $\mathbb{R}^n$, whose solution set we denote by $\widetilde{\mathcal{F}}(y)$. There exists a wide literature on the stability of $\widetilde{\mathcal{F}} : (\mathbb{R}^{n+1})^T \rightrightarrows \mathbb{R}^n$ when $(\mathbb{R}^{n+1})^T$ is equipped with the pseudometric of the uniform convergence (see [5] and references therein). By Lemma 2 in [3], given $y \in \mathrm{dom}\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}(y) = \mathrm{convbd}\widetilde{\mathcal{F}}(y)$ if and only if there exist indexes $s, u \in T$ and a point $x \in \mathbb{R}^n$ such that

$$(a_s' x)(a_u' x) < 0. \tag{9.5}$$

The set

$$Y := \left\{ y \in \left(\mathbb{R}^{n+1}\right)^T \mid \widetilde{\mathcal{F}}(y) = \mathrm{convbd}\widetilde{\mathcal{F}}(y) \right\}$$

contains almost all the elements of $(\mathbb{R}^{n+1})^T$ in a topological sense (Proposition 2 in[3]). Obviously, the restriction of $\widetilde{\mathcal{F}}$ to Y, say $\mathcal{F}$, satisfies (9.1). From (9.5) we get $\mathrm{dom}\mathcal{F} \subset \mathrm{int}Y$ and so the stability properties of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ coincide for every $y_0 \in \mathrm{dom}\mathcal{F}$, and the same is true for their respective boundary mappings. In particular, from the results on the stability of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{B}}$ at $y_0 \in \mathrm{dom}\mathcal{F}$ in [3] and [5], we can establish the following facts on the stability of $\mathcal{F}$ and $\mathcal{B}$ at $y_0 \in \mathrm{dom}\mathcal{F} = \mathrm{dom}\mathcal{B}$:

$\Diamond$ $\mathcal{F}$ is closed everywhere and the lsc and usc properties are independent of each other. Any of the previous examples show that condition (9.1) does not guarantee that $\mathcal{F}$ is closed everywhere.

$\Diamond$ If $\mathcal{B}$ is lsc at y_0, then $\mathcal{B}$ is closed at y_0, and the converse holds provided $\dim \mathcal{F}(y_0) = n$. The direct statement fails in Example 9.2.5 and the converse in Example 9.2.4 (although $\dim \mathcal{F}(y_0) = n$).

$\Diamond$ If $\mathcal{B}$ is usc at y_0, then $\mathcal{B}$ is closed at y_0, and the converse holds if $\mathcal{F}(y_0)$ is bounded. The direct statement is consequence of the closedness of $\mathcal{B}(y_0)$, whereas the converse statement fails in Example 9.2.4 (although $\mathcal{F}(y_0)$ is bounded).

$\Diamond$ If $\mathcal{B}$ is lsc at y_0, then $\mathcal{F}$ is lsc at y_0, and the converse holds. The direct statement holds in the previous examples but the converse fails in Example 9.2.3.

$\Diamond$ If $\mathcal{B}$ is usc at y_0, then $\mathcal{F}$ is usc at y_0 (just conjectured in [3]). The converse

statement fails in Example 9.2.3.
$\Diamond$ If $\mathcal{B}$ is closed at y_0, then $\mathcal{F}$ is closed at y_0 (trivial consequence of the closedness of $\mathcal{F}$ everywhere). The direct statement fails in Example 9.2.4 and the converse in Example 9.2.3.

Example 9.2.9 Let Y be a locally metrizable space such that $|Y| \geq 2$ and let us consider $y_0 \in Y$, a hyperplane $H \subset \mathbb{R}^n$ and the associated halfspaces H_- and H_+. Consider the mapping $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$ defined in the following terms

$$\mathcal{F}(y) := \begin{cases} H_-, & \textit{if } y = y_0, \\ H_+, & \textit{if } y \neq y_0. \end{cases}$$

It is evident that $\mathcal{F}$ is unstable in all senses at y_0 and it does not satisfy (9.1) whereas $\mathcal{B}$ is constant.

The consequence of the previous examples are:

1. The stability properties of $\mathcal{F}$ are independent of each other, with a unique (trivial) exception: if $\mathcal{F}$ is usc at y_0, then $\mathcal{F}$ is closed at y_0. The same statement is valid for $\mathcal{B}$.

2. $\mathcal{F}$ could inherit all the stability properties of $\mathcal{B}$ except closedness and continuity in the sense of Bouligand, provided condition (9.1) holds, but this condition does not guarantee the fulfillment of the converse statements which could require additional assumptions. These are the open problems to be solved in the next sections.

9.2.3 Lower semicontinuity

Proposition 9.2.1 *Let $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$ be a mapping satisfying (9.1) and let $y_0 \in \operatorname{dom}\mathcal{F}$. If $\mathcal{B}$ is lsc at y_0, then $\mathcal{F}$ is also lsc at y_0, and the converse statement holds if $\mathcal{F}$ is closed at y_0.*

Remark 9.2.2 Proposition 9.2.1 is still valid if the images of $\mathcal{F}$ are closed convex sets in a general normed space. Moreover, the first assertion certainly requires (9.1), but in the proof of the converse statement only the closedness and convexity of the images, and the closedness of $\mathcal{F}$ at y_0 are used.

9.2.4 Closedness

We know that the closedness of $\mathcal{B}$ is not sufficient to guarantee the closedness of $\mathcal{F}$ (Example 9.2.4). For this reason we require a closedness condition involving a family of closed-valued (but not convex-valued) mappings, $\mathcal{A}_\rho : Y \rightrightarrows \mathbb{R}^n$, with $\rho > 0$, such that

$$\mathcal{A}_\rho (y) := \{x \in \mathcal{B} (y) \mid \|x\| \leq \rho\} \cup \{x \in \mathcal{F} (y) \mid \|x\| = \rho\} .$$

We shall prove that, if $\mathcal{B}$ is usc at y_0 then $\mathcal{A}_\rho$ is closed at y_0, $\mathcal{A}_\rho$ being the boundary mapping of the truncated mapping of $\mathcal{F}$, $\mathcal{F}_\rho$.

Proposition 9.2.2 *Let $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$ be a mapping satisfying (9.1). Then the following statements hold:*

(i) $\mathcal{A}_\rho (y) = \mathrm{bd}\mathcal{F}_\rho (y)$ and $\mathcal{F}_\rho (y) = \mathrm{conv}\mathcal{A}_\rho (y)$ for all $\rho > 0$.

(ii) If $\mathcal{B}$ is usc at $y_0 \in Y$ and $\mathcal{A}_\rho (y_0) \neq \emptyset$, then $\mathcal{A}_\rho$ is closed at y_0.

Remark 9.2.3 As a consequence of Proposition 9.2.2(ii), if $\mathcal{B}$ is usc at $y_0 \in \mathrm{dom}\mathcal{F}$, then $\{\rho > 0 \mid \mathcal{A}_\rho \text{ is closed at } y_0\}$ is unbounded (this set contains the interval $[d (0_n, \mathcal{B} (y_0)) , +\infty[$) whereas the converse statement fails (see Example 9.2.1). Actually, the unboundedness of the set

$$\{\rho > 0 \mid \mathcal{A}_\rho \text{ is closed at } y_0\}$$

is an intermediate property between the upper semicontinuity and the closedness of $\mathcal{B}$. In fact, if $x_r \to x_0$, $y_r \to y_0$, and $x_r \in \mathcal{B}(y_r)$, $r = 1, 2, ...$, but $x_0 \notin \mathcal{B}(y_0)$, then taking $\rho > \|x_r\|$, $r = 0, 1, ...$, we have $x_r \in \mathcal{A}_\rho(y_r)$ for $r = 1, 2, ...$, but $x_0 \notin \mathcal{A}_\rho(y_0)$.

Proposition 9.2.3 *Let $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$ be a mapping satisfying (9.1) and let $y_0 \in \mathrm{dom}\mathcal{F}$. If the set $\{\rho > 0 \mid \mathcal{A}_\rho$ is closed at $y_0\}$ is unbounded, then $\mathcal{F}$ is closed at y_0.*

Example 9.2.3 shows that the converse statement is not true.

9.2.5 Upper semicontinuity

We state in this section that $\mathcal{F}$ always inherits the usc property from $\mathcal{B}$.

Proposition 9.2.4 *Let $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$ be a mapping satisfying (9.1), and suppose that $\mathcal{B}$ is usc at $y_0 \in Y$. Then $\mathcal{F}$ is also usc at y_0.*

Now, we give a positive answer to an open question in [3]. As in Example 9.2.8, we denote with $\widetilde{\mathcal{F}}$ the mapping which assigns to each linear inequality system in $\mathbb{R}^n$ with index set T, with $n \geq 2$ and $|T| \geq 2$, its corresponding solution set. We denote by $\widetilde{\mathcal{B}}$ the boundary mapping of $\widetilde{\mathcal{F}}$ (recall that $\widetilde{\mathcal{F}}$ does not satisfy condition (9.1)).

Corollary 9.2.1 *If $\widetilde{\mathcal{B}}$ is usc at $y_0 \in \mathrm{dom}\widetilde{\mathcal{F}}$, then $\widetilde{\mathcal{F}}$ is usc at y_0.*

9.2.6 Continuity

Whereas $\mathcal{F}$ inherits the continuity property from $\mathcal{B}$, the converse statement requires a boundedness assumption, which cannot easily be relaxed, as the final examples in the section show.

Proposition 9.2.5 *Let $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$ be a mapping satisfying (9.1) and consider $y_0 \in \mathrm{dom}\mathcal{F}$. If $\mathcal{B}$ is continuous in the sense of Berge at y_0, then $\mathcal{F}$ is also continuous in the sense of Berge at y_0, and the converse statement holds if $\mathcal{F}(y_0)$ is bounded.*

Let us consider again Example 9.2.7. Since $\mathcal{F}(y)$ is a polytope for all $y \in Y$, $\mathcal{B}$ is continuous (from now on in the sense of Berge) everywhere and so it is stable everywhere in all senses. The following example shows that the condition about the boundedness of the set $\mathcal{F}(y_0)$ cannot be suppressed in the last proposition.

Example 9.2.10 Consider the mapping $\mathcal{F} : [0, 1] \rightrightarrows \mathbb{R}^2$

$$\mathcal{F}(y) := \{x = (x_1, x_2) \mid x_1 \geq y x_2 \text{and } x_2 \geq y x_1\}.$$

It is evident that $\mathcal{F}$ satisfies (9.1), and the sets $\mathcal{F}(y)$, $y \in Y$, are all unbounded. It can easily be checked that $\mathcal{F}$ is continuous at 0, but $\mathcal{B}$ is not usc at 0.

The last example also shows that the boundedness of $\mathcal{F}(y_0)$ cannot be relaxed by replacing the condition "$\mathcal{F}$ is continuous at y_0 and $\mathcal{F}(y_0)$ is bounded" by the condition "$\mathcal{F}_\rho$ is continuous at y_0 for all ρ such that $\mathcal{F}(y_0) \cap B(0_n; \rho) \neq \emptyset$". The following example even shows that the converse statement in Proposition 9.2.5 is still false when we add, to the latter condition, the requirement that "$O^+\mathcal{F}(y) \subset O^+\mathcal{F}(y_0)$ for every y in a certain neighborhood of y_0", with O^+C denoting the recession cone of a closed convex set C.

Example 9.2.11 Consider the mapping $\mathcal{F} : [0,1] \rightrightarrows \mathbb{R}^2$

$$\mathcal{F}(y) := \mathbb{R}^2_+ \cap \{x = (x_1, x_2) \mid x_1 + y^2 x_2 \geq 2y\}.$$

It is evident that $\mathcal{F}$ satisfies (9.1), the images $\mathcal{F}(y)$, $y \in [0,1]$, are all unbounded, and $O^+\mathcal{F}(y) = \mathbb{R}^2_+$, for every $y \in [0,1]$.

Let us check that $\mathcal{F}$ is continuous at 0, which entails, by Lemma 9.2.2, Parts (ii) and (iv), that $\mathcal{F}_\rho$ is continuous at y_0 for all $\rho > 0$.

In fact, $\mathcal{F}$ is trivially usc at 0 because $\mathcal{F}(y) \subset \mathcal{F}(0) \equiv \mathbb{R}^2_+$, for every y.

Now, we prove that $\mathcal{F}$ is inner semicontinuous at 0, which is equivalent to the lower semicontinuity of $\mathcal{F}$ at this point. Hence, we have to verify that $\mathcal{F}(0) \subset \liminf_{r\to\infty} \mathcal{F}(y_r)$, for every sequence $\{y_r\}_{r=1}^\infty \subset Y$ converging to 0. To this aim, take an arbitrary, but fixed, $x_0 = (x_{01}, x_{02}) \in \mathcal{F}(0) \equiv \mathbb{R}^2_+$. If we define $x_r = (x_{r1}, x_{r2})$, with $x_{r2} = x_{02}$, $r = 1, 2, ..$, and

$$x_{r1} = \begin{cases} -(y_r)^2 x_{02} + 2y_r, & \text{if } x_{01} < -(y_r)^2 x_{02} + 2y_r, \\ x_{01}, & \text{otherwise,} \end{cases}$$

then it is clear that $x_r \in \mathcal{F}(y_r)$, $r = 1, 2, ...$, and $x_0 = \lim_{r\to\infty} x_r$.

Let us see, finally, that $\mathcal{B}$ is not usc at 0. It is evident that the open set

$$W := \mathbb{R}^2 \diagdown \{x = (x_1, x_2) \mid x_1 x_2 \geq 1 \text{and } x_1 \geq 0\},$$

contains $\mathcal{B}(0) = \{x = (x_1, x_2) \in \mathbb{R}^2_+ \mid x_1 x_2 = 0\}$, but $\mathcal{B}(y) \diagdown W \neq \emptyset$ for all $y \in]0,1]$ since, for these values of y, one observes $(y, 1/y) \in \mathcal{B}(y) \diagdown W$.

9.3 The relative boundary and extreme point set

9.3.1 Preliminaries

We say that $\mathcal{M} : Y \rightrightarrows \mathbb{R}^n$ is *locally convex* at $y_0 \in Y$ if there exists an open set $V \subset Y$, containing y_0, such that $\mathcal{M}(y)$ is convex for all $y \in V$. We shall use the following sufficient condition for $\mathcal{M}$ to be locally bounded.

Proposition 9.3.1 *Let $\mathcal{M} : Y \rightrightarrows \mathbb{R}^n$ and let $y_0 \in \operatorname{dom}\mathcal{M}$ such that $\mathcal{M}(y_0)$ is bounded and $\mathcal{M}$ is lsc, closed and locally convex at y_0. Then $\mathcal{M}$ is locally bounded and continuous at y_0.*

The condition of $\mathcal{M}$ being locally convex above is not superfluous as the following example shows.

Example 9.3.1 *If $Y = [0, 1]$ and $\mathcal{M} : Y \rightrightarrows \mathbb{R}$ is defined by $\mathcal{M}(y) = \{0, 1/y\}$ for $y \neq 0$ and $\mathcal{M}(0) = \{0\}$, then $\mathcal{M}$ is neither locally bounded nor continuous at $y_0 = 0$, in spite of $\mathcal{M}(y_0)$ being bounded and that $\mathcal{M}$ is lsc and closed at y_0.*

In order to characterize geometrically the conditions under which a given mapping with convex images is the convex hull of its associated mappings, let us recall that a halfflat is the intersection of a flat (also called affine manifold) with a closed halfspace which meets it, but does not contain it.

Proposition 9.3.2 *Given $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$, the following statements hold:*
(i) $\mathcal{F} = \operatorname{conv}\operatorname{bd}\mathcal{F}$ if and only if for every $y \in Y$, $\mathcal{F}(y)$ is a closed set which does not contain halfspaces.
(ii) $\mathcal{F} = \operatorname{conv}\operatorname{rbd}\mathcal{F}$ if and only if for every $y \in Y$, $\mathcal{F}(y)$ is a closed set which does not contain halfflats of the same dimension.
(iii) If $\mathcal{F}$ has closed images, then $\mathcal{F} = \operatorname{conv}\operatorname{ext}\mathcal{F}$ if and only if for every $y \in Y$ the set $\mathcal{F}(y)$ contains neither lines nor unbounded edges.

9.3.2 Lower semicontinuity

The following classical result ([11], Proposition 2.6) is an extension to arbitrary mappings of the direct statement of Proposition 9.2.1 (where $\mathcal{A}$ =bd$\mathcal{F}$).

Theorem 9.3.1 *If $\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$ is lsc at $y_0 \in \mathrm{dom}\mathcal{A}$, then* conv$\mathcal{A}$ *is also lsc at y_0.*

The converse statement of Proposition 1 9.2.1 establishes that, if $\mathcal{F}$ =conv bd $\mathcal{F}$ is lsc and closed at $y_0 \in \mathrm{dom}\mathcal{F}$, then bd $\mathcal{F}$ is lsc at y_0. The next two results are counterparts of this statement for rbd$\mathcal{F}$ and ext$\mathcal{F}$ (instead of bd $\mathcal{F}$). Example 9.2.4, where bd $\mathcal{F}$ =rbd$\mathcal{F}$ = ext$\mathcal{F}$, shows that the closedness of $\mathcal{F}$ is not superfluous in these results. The following example shows that, in general, if conv$\mathcal{A}$ is lsc and closed at y_0, then $\mathcal{A}$ is not necessarily lsc at y_0. Accordingly, the proofs must appeal to the specific properties of the sets rbd$\mathcal{F}(y)$ and ext$\mathcal{F}(y)$.

Example 9.3.2 Let $\mathcal{A} : \mathbb{R} \rightrightarrows \mathbb{R}$ such that

$$\mathcal{A}(y) = \begin{cases} \{-1, 0, 1\}, & y = 0, \\ \{-1, 1\}, & y \neq 0. \end{cases}$$

It is easy to see that conv$\mathcal{A}$ is constant (so that it is continuous and closed) whereas $\mathcal{A}$ is not lsc at $y_0 = 0$.

Theorem 9.3.2 *Let $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$ such that $\mathcal{F}$ =*convrbd $\mathcal{F}$ *and $\mathcal{F}$ is lsc and closed at $y_0 \in$* dom$\mathcal{F}$. *Then* rbd$\mathcal{F}$ *is lsc at y_0.*

Theorem 9.3.3 *Let $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$ such that $\mathcal{F}$ =*convextr $\mathcal{F}$ *and $\mathcal{F}$ is lsc and closed at $y_0 \in$* dom$\mathcal{F}$. *Then* extr$\mathcal{F}$ *is lsc at y_0.*

9.3.3 Upper semicontinuity and closedness

In contrast with lower semicontinuity, the closedness of a set-valued mapping $\mathcal{A}$ is not inherited by conv$\mathcal{A}$ (even though $\mathcal{A}$ =bd$\mathcal{F}$, rbd$\mathcal{F}$, ext$\mathcal{F}$, as Example 9.2.4 shows). On the other hand, Proposition 9.2.4 establishes that, if bd$\mathcal{F}$

is usc at y_0, then $\mathcal{F}$ is usc at y_0. In this section we shall prove that a similar statement holds for rbd$\mathcal{F}$, but not for ext$\mathcal{F}$ even though ext$\mathcal{F}$ is either locally bounded or closed (nevertheless according to the next Theorem 9.3.4, these two properties together entail the upper semicontinuity and the closedness of $\mathcal{F}$).

Example 9.3.3 Let $\mathcal{E} : Y \rightrightarrows \mathbb{R}^2$, where $Y = [2, +\infty[$ and

$$\mathcal{E}(y) = \left\{ x \in \mathbb{R}^2 \mid \|x\| = 1, x_1 < y^{-1} \right\} \cup \{(y, 0)\} \text{ for all } y \in Y.$$

It is easy to see that $\mathcal{E}$ is locally bounded and continuous but not closed at $y_0 = 2$, and that it is the extreme points set mapping of $\mathcal{F} = \text{conv}\mathcal{E}$. We shall prove that $\mathcal{F}$ is not usc at y_0. Let

$$W := \left\{ x \in \mathbb{R}^2 \mid \sqrt{3}\,|x_2| < 2 - x_1, x_1 < 2 \right\} \cup B\left((2,0)\,; \frac{1}{2} \right),$$

$\mathcal{F}(y_0) \subset W$. If $y > 2$, then $\overline{x} = \left(1, \frac{1}{\sqrt{3}}\right) \in \mathcal{F}(y) \setminus W$. Observe also that $\mathcal{F}$ cannot be closed at y_0 (because $\mathcal{F}(y_0)$ is not closed).

Example 9.3.4 Let $\mathcal{E} : \mathbb{R} \rightrightarrows \mathbb{R}^3$ such that

$$\mathcal{E}(y) = \left\{ (x_1, x_2, 0) \in \mathbb{R}^3 \mid x_2 = x_1^2 \right\} \cup \{(0, 0, y)\} \text{ for all } y \in \mathbb{R}.$$

As in the previous example, $\mathcal{E} = \text{ext}\mathcal{F}$ for $\mathcal{F} = \text{conv}\mathcal{E}$ and $\mathcal{E}$ is continuous at $y_0 = 0$, but now $\mathcal{E}$ is also closed and $\mathcal{E}(y_0)$ is unbounded. In order to prove that $\mathcal{F}$ is not usc at y_0, let us consider the convex plane set $C := \{x \in \mathbb{R}^2 \mid x_2 \geq x_1^2\}$ and the open set

$$W := \mathbb{R}^3 \setminus \left\{ x \in \mathbb{R}^3 \mid x_3 \geq x_2^{-1}, x_2 > 0 \right\}.$$

Obviously, $\mathcal{F}(y_0) = C \times \{0\} \subset W$. Moreover, if $y > 0$ and $y > 4/r^2$ for $0 \neq r \in \mathbb{R}$, we have

$$\left(0, \frac{r^2}{2}, \frac{y}{2}\right) = \frac{1}{2}(0, 0, y) + \frac{1}{4}\left[\left(-r, r^2, 0\right) + \left(r, r^2, 0\right) \right] \in \mathcal{F}(y) \setminus W,$$

so that $\mathcal{F}(y) \nsubseteq W$. Hence $\mathcal{F}$ is not usc at y_0.
Finally, we show that $\mathcal{F}$ is closed at y_0. Let $y_r \to y_0$ and $x^r \to x^0$ such that $x^r \in \mathcal{F}(y_r)$, $r = 1, 2, ...$ Since $\mathcal{F}(y_r) = \operatorname{conv}[(C \times \{0\}) \cup \{(0,0,y_r)\}]$, for any $r \in \mathbb{N}$, we can write

$$x^r = \lambda_r (c^r, 0) + (1-\lambda_r)(0,0,y_r) = (\lambda_r c^r, (1-\lambda_r) y_r), c^r \in C, 0 \leq \lambda_r \leq 1.$$

Observe that $c^r \in C$ and $(0,0) \in C$ entail $\lambda_r c^r \in C$. On the other hand, $x_3^r = (1-\lambda_r) y_r \in \operatorname{conv}\{0, y_r\}$. Taking limits we get $x^0 = \lim_r x^r \in C \times \{0\} = \mathcal{F}(y_0)$.

The next result is a reformulation of a well-known result ([1], Lemma 1.1.9), taking into account the mentioned equivalence between closedness and outer semicontinuity. It can be seen as an extension of the first part of Proposition 9.2.3, which was also based on Carathéodory's theorem.

Theorem 9.3.4 *If* $\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$ *is closed and locally bounded at* $y_0 \in \operatorname{dom}\mathcal{A}$*, then* $\operatorname{conv}\mathcal{A}$ *is closed and usc at* y_0*.*

Observe that it is not possible to replace in the Theorem 9.3.4 above the condition "$\mathcal{A}$ is closed and locally bounded at y_0" by just "$\mathcal{A}$ is closed and usc at y_0" (recall Example 9.3.4).

Given two set-valued mappings $\mathcal{M}, \mathcal{N} : Y \rightrightarrows \mathbb{R}^n$, we say that $\mathcal{M}$ is *contained* in $\mathcal{N}$ (in brief, $\mathcal{M} \subset \mathcal{N}$) *locally at* y_0 if there exists an open set $V \subset Y$, containing y_0, such that $\mathcal{M}(y) \subset \mathcal{N}(y)$ for all $y \in V$. We also define the *closure* of $\mathcal{M}$ as the mapping $\operatorname{cl}\mathcal{M} : Y \rightrightarrows \mathbb{R}^n$ such that $(\operatorname{cl}\mathcal{M})(y) = \operatorname{cl}\mathcal{M}(y)$ for all $y \in Y$.

Corollary 9.3.1 *Let* $\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$ *and let* $y_0 \in \operatorname{dom}\mathcal{A}$ *such that* $\mathcal{A}(y_0)$ *is bounded and* $\mathcal{A}$ *is usc at* y_0*. Then each of the following conditions guarantees that* $\operatorname{conv}\mathcal{A}$ *is closed and usc at* y_0*:*
(i) $\mathcal{A}(y_0)$ *is closed.*
(ii) $\operatorname{cl}\mathcal{A} \subset \operatorname{conv}\mathcal{A}$ *locally at* y_0*.*

The boundedness assumption in Corollary 9.3.1 is not superfluous even for the extreme points set mapping (recall again Example 9.3.4, where (i) holds).

Now, we give a condition that assures that if $\mathcal{A}$ is usc at y_0, then conv$\mathcal{A}$ is usc at y_0 as well.

Proposition 9.3.3 *Let* $\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$ *and let* $y_0 \in \mathrm{dom}\mathcal{A}$ *such that*

$$\mathrm{rbdconv}\mathcal{A} \subset \mathcal{A} \subset \mathrm{convrbdconv}\mathcal{A}$$

locally at y_0 *and* conv$\mathcal{A}$ *is closed at* y_0*. If* $\mathcal{A}$ *is usc at* y_0*, then* conv$\mathcal{A}$ *is usc at* y_0*.*

Given $\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$ and $\rho > 0$, we denote by $\mathcal{A}_\rho$ and by $(\mathrm{conv}\mathcal{A})_\rho$ the truncated mappings of $\mathcal{A}$ and conv$\mathcal{A}$, respectively, with radius ρ. We also define the mapping $\mathcal{A}^\rho : Y \rightrightarrows \mathbb{R}^n$ such that

$$\mathcal{A}^\rho (y) = \mathcal{A}_\rho (y) \cup \{x \in \mathrm{conv}\mathcal{A}(y) \mid \|x\| = \rho\}.$$

It is easy to prove that, if $\mathcal{F} = \mathrm{convrbd}\mathcal{F}$ ($\mathcal{F} = \mathrm{convbd}\mathcal{F}$), and $\mathcal{A} = \mathrm{rbd}\mathcal{F}$ ($\mathcal{A} = \mathrm{bd}\mathcal{F}$, respectively), then $(\mathrm{conv}\mathcal{A})_\rho = \mathrm{conv}\mathcal{A}^\rho$.

Lemma 9.3.1 *Let* $\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$ *and let* $y_0 \in \mathrm{dom}\mathcal{A}$ *such that* $\mathcal{A}(y_0)$ *and* conv
$\mathcal{A}(y_0)$ *are closed and* $\mathcal{A}$ *is usc at* y_0*. Then* $\{\rho > 0 \mid \mathcal{A}^\rho$ *is closed at* $y_0\}$ *is unbounded.*

Lemma 9.3.2 *Let* $\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$ *such that* $(\mathrm{conv}\mathcal{A})_\rho = \mathrm{conv}\mathcal{A}^\rho$ *for all* $\rho > 0$ *sufficiently large and let* $y_0 \in \mathrm{dom}\mathcal{A}$ *such that* $\{\rho > 0 \mid \mathcal{A}^\rho$ *is closed at* $y_0\}$ *is unbounded. Then* conv$\mathcal{A}$ *is closed at* y_0*.*

Proposition 9.3.4 *Let* $\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$ *such that* $(\mathrm{conv}\mathcal{A})_\rho = \mathrm{conv}\mathcal{A}^\rho$ *for all* $\rho > 0$ *sufficiently large and let* $y_0 \in \mathrm{dom}\mathcal{A}$ *such that* $\mathcal{A}(y_0)$ *is closed,*

$$\mathrm{rbdconv}\mathcal{A} \subset \mathcal{A} \subset \mathrm{convrbdconv}\mathcal{A}$$

locally at y_0 *and* $\mathcal{A}$ *is usc at* y_0*. Then* conv$\mathcal{A}$ *is usc at* y_0*.*

Theorem 9.3.5 *Let* $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$ *be such that* $\mathcal{F} = \mathrm{convrbd}\ \mathcal{F}$ *and* $\mathrm{rbd}\mathcal{F}$ *is usc at* $y_0 \in \mathrm{dom}\mathcal{F}$. *Then* $\mathcal{F}$ *is usc at* y_0.

The last four results are also valid replacing everywhere "rbd" with "bd". The final example illustrates the results in Sections 3 and 4 and shows that there is no usc counterpart for Theorems 9.3.2 and 9.3.3.

Example 9.3.5 Let us identify the complex field $\mathbb{C}$ with $\mathbb{R}^2$ and let us take as Y the set of polynomials of degree $q \in \mathbb{N}$ (fixed) with complex coefficients equipped with the Euclidean distance on $\mathbb{R}^{2q+2}$. Given $y \in Y$, we denote by $\mathcal{A}(y)$ its set of complex zeros and by $\mathcal{F}(y)$ its convex hull, i.e., the polytope $\mathcal{F}(y) = \mathrm{conv}\mathcal{A}(y)$. By the fundamental theorem of algebra, $\mathcal{A}(y) \neq \emptyset$ for all $y \in Y$, so that $\mathrm{dom}\mathcal{A} = Y$. Let us denote by $\mathcal{B}$, $\mathcal{R}$ and $\mathcal{E}$ the boundary mapping, the relative boundary mapping and the extreme points set mapping of $\mathcal{F}$, respectively. By Proposition 9.3.2, we have

$$\mathcal{F} = \mathrm{conv}\mathcal{B} = \mathrm{conv}\mathcal{R} = \mathrm{conv}\mathcal{E}.$$

$\mathcal{A}$ is lsc and usc as a consequence of a well-known consequence of Rolle's Theorem for complex polynomials (see, e.g., [10]) and, since it has closed images, it is also closed. By Theorem 9.3.1 and Corollary 9.3.1, $\mathcal{F}$ is also lsc, usc and closed. Consequently, $\mathcal{B}$, $\mathcal{R}$ and $\mathcal{E}$ are lsc by Propositions 1 in [4] and Theorems 9.3.2 and 9.3.3 in this paper (the direct proofs of these statements are rather involved). Now, we show that $\mathcal{R}$ and $\mathcal{E}$ are neither usc nor closed if $q = 3$.
Let $y_0 = x^3 + x$, with $\mathcal{A}(y_0) = \{0, \pm i\}$, and let $y_r = x^3 - \frac{2}{r}x^2 + \left(1 + \frac{1}{r^2}\right)x$, with $\mathcal{A}(y_r) = \left\{0, \frac{1}{r} \pm i\right\}$, $r = 1, 2, ..$ Obviously, $y_r \to y_0$. Taking the constant sequence $x_r = 0$, $r = 1, 2, ..$ we have $x_r \in \mathcal{E}(y_r) \subset \mathcal{F}(y_r)$ for all r, whereas $0 \notin \mathcal{E}(y_0) = \mathcal{R}(y_0) = \{\pm i\}$. Thus neither $\mathcal{R}$ nor $\mathcal{E}$ is closed (usc) at y_0.

9.4 References

[1] J.-P. Aubin, H. Frankowska, Set-Valued Analysis, Birkhäuser, Boston, 1990.

[2] B. Bank, J. Guddat, D. Klatte, B. Kummer, K. Tammer, Non-Linear Parametric Optimization, Birkhäuser Verlag, Basel, 1983.

[3] M.A. Goberna, M. Larriqueta, V.N. Vera de Serio, On the stability of the boundary of the feasible set in linear optimization, Set-valued Analysis, Vol. 11 (2003), pp. 203-223.

[4] M.A. Goberna, M. Larriqueta, V.N. Vera de Serio, On the stability of the extreme point set in linear optimization, SIAM J. Optimization, Vol. 15, (2005), pp. 1155-1169.

[5] M.A. Goberna, M.A. López, Linear Semi-Infinite Optimization, Wiley, Chichester, 1998.

[6] M.A. Goberna, M.A. López, M.I. Todorov, Stability for linear inequality systems II: Upper semicontinuity of the solution set mapping, SIAM J. Optim. 7 (1997), pp. 1138-1151.

[7] M.A. Goberna, M.A. López, M.I. Todorov, On the stabiliy of closed-convex-valued mappings and the associated boundaries. JMAA, Vol. 306, Issue 2, (2005), pp. 502-512.

[8] M.A. Goberna, M.I. Todorov, V.N. Vera de Serio, On the stability of convex-valued mappings and their relative boundary and extreme point set mappings, SIAM J. on Optimization, Vol. 17, No. 1, pp. 147-158, 2006.

[9] S. Hu, N.S. Papageorgiou, Handbook of Multivalued Analysis. Volume I: Theory, Kluwer, Dordrecht (NL), 1997.

[10] J.E. Marsden, M.J. Hoffman, Basic Complex Analysis, Freeman, New York, 1987.

[11] E. Michael, Continuous selections (I), The Annals of Mathematics, 2nd Ser. 63 (1956), 361-382.

[12] R.T. Rockafellar, R.B. Wets, Variational Analysis, Springer-Verlag, New York, 1998.

10

Some estimates for one-sided trigonometric approximation

Jorge Bustamante [1]

Abstract: We present some estimates for one-sided trigonometric approximation in L_p spaces. We pay attention to the constants related with the Jackson type inequalities. The proof are obtained by constructing some special Steklov type functions.

Keywords: One-sided approximation, Rate of convergence, Steklov type functions and Estimates of constants

Classification: MSC 41A29, 41A17, 41A25 and 41A44

10.1 Introduction

In this paper we use the following notations. Let L_p $(1 \leq p < \infty)$ be the space of all 2π-periodic real-valued functions such that

$$\|f\|_p = \left(\int_{-\pi}^{\pi} \mid f(t) \mid^p dt \right)^{1/p} < \infty.$$

[1]Faculta de Ciencias Físico Matemáticas,Benemérita Universidad Autónoma de Puebla, jbusta@fcfm.buap.mx

Moreover, let C be the family of all 2π-periodic continuous function $f : \mathbb{R} \to \mathbb{R}$ with the sup norm $\|f\|_\infty = \sup\{| f(t) | : x \in [0, 2\pi]\}$.

We denote by $\mathbb{T}_n$ be the family of all trigonometric polynomials of degree not greater than n.

The notation $f \geq g$ ($f \leq g$) will mean that $f(x) \geq g(x)$ ($f(x) \leq g(x)$) for all $x \in [0, 2\pi]$.

As usual, for $f \in L_p$ (C), the best trigonometric approximation of degree n is defined by

$$E_n(f)_p \ = \ \inf\{ \ \|f - T_n\|_p \ : \ T_n \in \mathbb{T}_n \ \}.$$

Let $\mathcal{R}[0, 2\pi]$ be the set of all 2π-periodic Riemann integrable functions (for approximation in C we assume that $\mathcal{R}[0, 2\pi] = C$). The best one-sided approximation, in the L_p-metric, of a function $f \in \mathcal{R}[0, 2\pi]$ by trigonometric polynomials of order n is defined by

$$E^*_n(f)_p \ = \ \inf\{ \ \|t_n - T_n\|_p \ : \ t_n, T_n \in \mathbb{T}_n, \ \ t_n \leq f \leq T_n \ \}.$$

For continuous functions the best one-sided approximation $E^*_n(f)_\infty$ is defined analogously.

It is easy to see that, for $f \in C$ and $n \in \mathbb{N}$,

$$E_n(f)_\infty \ \leq \ E^*_n(f)_\infty \ \leq \ 2E_n(f)_\infty.$$

Thus, the orders of the best approximation and the best one-sided approximation coincide. In L_p spaces the situation is different. Ivanov [6] showed that, for $1 \leq p < \infty$ and any $r \in \mathbb{N}_0$, there exists an r-times continuously differentiable function $f \in \mathcal{R}[0, 2\pi]$ for which

$$\lim_{n\to\infty} \sup \ \frac{E^*_n(f)_p}{E_n(f)_p} \ = \infty \, .$$

Jianli and Songping obtained a stronger result [7]. For $1 < p < \infty$, there exists $f \in C$ for which

$$\lim_{n\to\infty} \sup \ \frac{E^*_n(f)_p}{\omega(f, 1/n)_p} \ = \infty \, , \tag{10.1}$$

where $\omega(f,t)_p$ is the usual modulus of continuity. Recall that, for $f \in L_p$, $r \in \mathbb{N}$ and $t > 0$, the *usual modulus of smoothness of order* r is defined by

$$\omega_r(f,t)_p = \sup\{\|\Delta_h^r f\|_p : |h| \le t\} \tag{10.2}$$

where

$$\Delta_h^r f(x) = \sum_{k=0}^{r}(-1)^r \binom{r}{k} f(x+kh).$$

For $f \in C$ we change the L_p norm by the uniform one.

From (10.1) we know that the usual moduli of smoothness are not the adequate one for direct results in one-sided approximation. That is the reason why the Bulgarian school of approximation proposed another modulus. Let us present the definition.

The local modulus of continuity of order r of a function $f \in \mathcal{R}[0,2\pi]$ at a point x is defined by

$$\omega_r(f,x,t) = \sup\left\{\left|\Delta_h^k f(s)\right| : h > 0,\ s, s+rh \in [x-rt, x+rt]\right\}. \tag{10.3}$$

Now define

$$\tau_r(f,t)_p = \|\omega_r(f,\cdot,t)\|_p \qquad (\tau_k(f,t)_\infty = \|\omega_r(f,\cdot,t)\|_\infty).$$

When $r = 1$ we simply write $\tau(f,t)_p$. These moduli are well defined whenever f is a bounded measurable function. But here we only work with the class $\mathcal{R}[0,2\pi]$, because it is known that $\lim_{t\to 0+}\tau(f,t)_p = 0$ if and only if $f \in \mathcal{R}[0,2\pi]$ (see [4]).

Moduli τ_r turn out to be very useful in order to characterize the orders of one-sided approximations of functions in L_p-metric. For instance Popov and Andreev [9] verified that, for $1 \le p \le \infty$ and $r \in \mathbb{N}$, there exists a constant C_r such that, for all $f \in M$ and $n \in \mathbb{N}$

$$E_n^*(f)_p \le C_r\, \tau_r\left(f, \frac{\pi}{n}\right)_p. \tag{10.4}$$

In [8] Popov presented the converse result: for $1 \le p \le \infty$ and $r \in \mathbb{N}$, there exists a constant C_r such that, for all $f \in M$ and $n \in \mathbb{N}$

$$\tau_r\left(f, \frac{\pi}{n}\right)_p \le \frac{C_r}{n^r} \sum_{k=0}^{n} (k+1)^{r-1} E_k^*(f)_p.$$

In this paper we will present another proof of (10.4) in the case $r = 1$. We remark that in this case a proof of (10.4) was given in [1] using splines. Here we follow a different approach. We construct some Steklov type functions that, in our opinion, are the appropriated one to analyze one-sided approximation. These functions are studied in Section 2. In Section 3 we present upper estimates for the best one-sided approximation. The ideas used here can be generalized to obtain estimates in terms of the modulus τ_r, with $r > 1$. But we do not know if these extensions will give place to good constants.

10.2 Steklov type functions

In [10] we find the following assertion, If $f \in \mathcal{R}[0, 2\pi]$, $u, t \in [0, \pi]$ and $r \in \mathbb{N}$, then

$$| f(x \pm t) - f(x) | \leq \omega_r(f, x \pm u, t) + \omega_r(f, x \pm t, u).$$

We will use a simpler (easy to prove) inequality.

Proposition 10.2.1 *If $f \in \mathcal{R}[0, 2\pi]$ and $x, s \in \mathbb{R}$, then*

$$| f(x \pm s) - f(x) | \leq \omega(f, x + s/2, s).$$

Proof. It follows for the definition of the local modulus of continuity that

$$| f(x+s) - f(x) | = \left| f\left(x + \frac{s}{2} + \frac{s}{2}\right) - f\left(x + \frac{s}{2} - \frac{s}{2}\right) \right| \leq \omega(f, x+s/2, s). \quad \square$$

For a function $f \in \mathcal{R}[0, 2\pi]$ and $h > 0$ define the Steklov type functions

$$f_{\pm}(x) = \frac{1}{h^2} \int_0^h \int_h^{2h} [f(x+s) \pm \omega(f, x+s/2, t)] dt ds. \tag{10.5}$$

Proposition 10.2.2 *If $f \in \mathcal{R}[0, 2\pi]$, $h > 0$ and the functions $f_{\pm}$ are defined by* (10.5)*, then*

(i) $f_\pm \in \mathcal{R}[0,2\pi]$ *and* $f_\pm$ *are derivable. Moreover,*

$$f'_\pm(x) = \frac{1}{h}[f(x+h)-f(x)] \pm \frac{1}{h^2}\int_h^{2h} [w(f,x+h/2,t)-w(f,x,t)]dt. \quad (10.6)$$

(ii) *For each* $x \in [0,2\pi]$

$$f_-(x) \le f(x) \le f_+(x).$$

(iii) *For* $1 \le p < \infty$ *one has*

$$\|f - f_\pm\|_p \le \tau(f,2h)_p \quad (10.7)$$

$f'_\pm \in L_p$ *and*

$$\|f'_\pm\|_p \le \frac{3}{h}\tau(f,2h)_p. \quad (10.8)$$

Proof. (i) Notice that, for any $f \in \mathcal{R}[0,2\pi]$, the local modulus of continuity (10.3) is a bounded function 2π-periodic with respect to x. Thus $f_\pm \in \mathcal{R}[0,2\pi]$. It follows from the known results related with Steklov type functions, that, if $g \in L_1$, $u > 0$ and

$$G_u(x) = \frac{1}{u}\int_0^u g(x+s)ds,$$

then g is derivable and

$$G'_u(x) = \frac{1}{u}\,[g(x+u) - g(x)].$$

In particular, if we consider the function

$$g(x) = \frac{1}{h}\int_h^{2h} \omega(f,x,t)dt,$$

the $G_{h/2}$ is derivable and

$$G'_{h/2}(x) = \frac{2}{h}\,[g(x+h/2) - g(x)] = \frac{2}{h^2}\int_h^{2h} [\omega(f,x+h/2,t) - \omega(f,x,t)]dt.$$

But

$$G_{h/2}(x) \;=\; \frac{2}{h^2}\int_0^{h/2}\int_h^{2h}\omega(f,x{+}s,t)dtds \;=\; \frac{1}{h^2}\int_0^{h/2}\int_h^{2h}\omega(f,x{+}s/2,t)dtds.$$

Thus, assertion (i) is proved.

(ii) From Prop. 10.2.1 one has

$$\begin{aligned} f(x)-f_+(x) \;&=\; \frac{1}{h^2}\int_0^{h}\int_h^{2h}[f(x)-f(x+s)-\omega(f,x+s/2,t)]dtds \\ &\le\; \frac{1}{h^2}\int_0^{h}\int_h^{2h}[\omega(f,x+s/2,s)-\omega(f,x+s/2,t)]dtds \;\le 0. \end{aligned}$$

With similar arguments the inequality $f_-(x)\le f(x)$ is proved.

(iii) For the assetion of (iii) we present a proof for a fixed $1<p<\infty$ (the case $p=1$ follows analogously). As usual, take q such that $1/p+1/q=1$. Using twice Hölder inequality and considering that the local modulus of continuity (10.3) is a 2π-periodic function, we obtain

$$\begin{aligned} (h^2\|f_+-f\|_p)^p &\le \int_0^{2\pi}\left(\int_0^{h}\int_h^{2h}[\omega(f,x+s/2,t)-\omega(f,x+s/2,s)]dtds\right)^p dx \\ &\le\; h^{p/q}\int_0^{2\pi}\left(\int_0^{h}\left(\int_h^{2h}\omega^p(f,x+s/2,t)dt\right)^{1/p}ds\right)^p dx \\ &\le\; h^{2p/q}\int_0^{2\pi}\int_0^{h}\int_h^{2h}\omega^p(f,x+s/2,t)dtdsdx \\ &=\; h^{2p/q}\int_0^{h}\int_h^{2h}\int_0^{2\pi}\omega^p(f,x+s/2,t)dxdtds \\ &=\; h^{2p/q}\int_0^{h}\int_h^{2h}\int_0^{2\pi}\omega^p(f,x,t)_p\,dxdtds \\ &=\; h^{2(p-1)}\int_0^{h}\int_h^{2h}\tau^p(f,t)_p\,dtds \le\; h^{2p}\,\tau^p(f,2h)_p. \end{aligned}$$

For $\|f - f_-\|_p$ the proof follows analogously,

Finally, in order to estimate $f'_\pm$, we use the representation of the derivative given in (i). It is known that (see [1], Lemma 4), for any $f \in \mathcal{R}[0, 2\pi]$

$$\omega\,(f, h)_p \,\le\, \tau\,(f, h)_p.$$

Therefore

$$\|f(\cdot + h) - f(\cdot)\|_p \,\le\, \omega(f, h)_p \,\le\, \tau\,(f, h)_p. \tag{10.9}$$

On the other hand

$$\left(\int_0^{2\pi}\left|\int_h^{2h}[\omega(f, x + h/2, t) - \omega(f, x, t)]dt\right|^p\,dx\right)^{1/p}$$

$$\le \left(\int_0^{2\pi}\left|\int_h^{2h}\omega(f, x + h/2, t)dt\right|^p\,dx\right)^{1/p} + \left(\int_0^{2\pi}\left|\int_h^{2h}\omega(f, x, t)dt\right|^p\,dx\right)^{1/p}$$

$$\le h^{1/q}\left(\int_0^{2\pi}\int_h^{2h}\omega^p(f, x + h/2, t)dt\,dx\right)^{1/p} + h^{1/q}\left(\int_0^{2\pi}\int_h^{2h}\omega^p(f, x, t)dt\,dx\right)^{1/p}$$

$$= h^{1/q}\left(\int_h^{2h}\int_0^{2\pi}\omega^p(f, x + h/2, t)dx\,dt\right)^{1/p} + h^{1/q}\left(\int_h^{2h}\int_0^{2\pi}\omega^p(f, x, t)dx\,dt\right)^{1/p}$$

$$= 2h^{1/q}\left(\int_h^{2h}\int_0^{2\pi}\omega^p(f, x, t)dx\,dt\right)^{1/p} = 2h^{1/q}\left(\int_h^{2h}\tau^p(f, t)_p\,dt\right)^{1/p}$$

$$\le\, 2h\;\tau\,(f, 2h)_p. \tag{10.10}$$

From (10.6), (10.9) and (10.10), we obtain (10.8). □

10.3 Estimates for one-sided approximation

In this section we present improved versions of some of the results given by Andreev, Popov and Sendov in [1]. We will consider the quantities

$$E_n^+(f)_p \,=\, \inf\{\ \|T_n - f\|_p \,:\, T_n \in \mathbb{T}_n,\ \ f \le T_n\ \}$$

and

$$E_n^-(f)_p = \inf\{ \|f - t_n\|_p : t_n, \in \mathbb{T}_n, \ t_n \leq f \}$$

instead of $E_n^*(f)_p$. Of course

$$E_n^*(f)_p \leq E_n^+(f)_p + E_n^-(f)_p,$$

$$E_n^+(f)_p \leq E_n^*(f)_p \quad \text{and} \quad E_n^-(f)_p \leq E_n^*(f)_p.$$

For the proof we need some results of Ganelius, Babenko and Ligun.

For $r \in \mathbb{N}$, let $W^r L_p$ be the class of all 2π-periodic functions f such that $f^{(r-1)}$ ($f^{(0)} = f$) is absolutely continuous and $\|f^{(r)}\|_p \leq 1$.

Theorem 10.3.1 [2] *If $p > 1$, $f \in W^r L_p$ and $n \geq r$,*

$$E_n^{\pm}(f)_p \leq \frac{2\pi}{3\, n^r} \|f^{(r)}\|_p. \tag{10.11}$$

The original statement of Th. 10.3.1 asserts that

$$\sup \left\{ E_n^-(f)_p : f \in W^r L_p \text{ and } \|f^{(r)}\|_p \leq 1 \right\} = \mathcal{O}(n^{-r}),$$

as $n \to \infty$. But a carefully reading of the proof shows that the result can written as we did above.

For the case $p = 1$ the corresponding estimate was previously published by Ganelius. In fact, the proof Th.10.3.1 us the Ganelius result. We denote by $V_a^b(f)$ the total variation of f on the interval $[a, b]$.

Theorem 10.3.2 [5] *For each $r \in \mathbb{N}_0$, there exists a constant $C(r) \leq 1/2$ such that, if $V_0^{2\pi}(f^{(r)}) < \infty$ and $n \geq r$, there exists $U_{n-1} \in \mathbb{T}_{n-1}$ such that $U_{n-1}(x) \geq f(x)$ and*

$$\|U_{n-1} - f\|_1 \leq \frac{2\pi C(r)}{n^{r+1}} V_0^{2\pi}(f^{(r)}).$$

In our next result we provide a estimate for the constant C in Th. 3 of [1]. In this case our proof is different from the one given in [1], where piecewise linear function were used as an intermediate approximation. Here we use the Steklov type functions presented above instead of piecewise linear function.

Theorem 10.3.3 *If* $1 \leq p < \infty$, $f \in \mathcal{R}[0, 2\pi]$ *and* $n \in \mathbb{N}$, *then*

$$E_n^{\pm}(f)_p \ \leq \ 3\, \tau \left(f, \frac{2\pi}{n} \right)_p .$$

Proof. Let $f_{\pm}$ be given as in Prop. 10.2.2 with $h = \pi/n$. It is easy to see that

$$E_n^{+}(f)_p \ \leq \ E_n^{+}(f_+)_p + \|f_+ - f\|_p$$

and

$$E_n^{-}(f)_p \ \leq \ E_n^{-}(f_-)_p + \|f - f_-\|_p.$$

From (10.11), (10.7) and (10.8) one has

$$E_n^{\pm}(f)_p \ \leq \ \frac{2\pi}{3n} \|f_{\pm}\|_p + \ \tau \left(f, \frac{2\pi}{n} \right)_p \leq 3\, \tau \left(f, \frac{2\pi}{n} \right)_p . \qquad \square$$

We need the following proposition taken from [3].

Proposition 10.3.1 *Assume* $1 \leq p < \infty$ *and* $r \in \mathbb{N}$. *If* $f \in W_p^r[0, 2\pi]$ *and there exists* $T \in \mathbb{T}_n$ *such that* $T \geq f^{(r)}$ *for* $x \in [0, 2\pi]$, *then there exists* $R \in \mathbb{T}_n$ *such that* $R \geq f$ *and*

$$\left(\int_0^{2\pi} |R(x) - f(x)|^p \, dx \right)^{1/p} \ \leq \ \left(\frac{\pi}{n} \right)^r \ \|f^{(r)} - T\|_p \, . \tag{10.12}$$

If there exists $T \in \mathbb{T}_n$ *such that* $T \leq f^{(r)}$, *an analogous result holds with a polynomial* R *satisfying* $R \leq f$.

Theorem 10.3.4 *Fix* $1 \leq p < \infty$. *If* $f \in \mathcal{R}[0, 2\pi]$ *has a derivative* $f' \in \mathcal{R}[0, 2\pi]$, *then for each* $n \in \mathbb{N}$, *one has*

$$E_n^{*}(f)_p \ \leq \ \frac{6\pi}{n}\, \tau \left(f', \frac{2\pi}{n} \right)_p .$$

Proof. From Th. 10.3.3, for each $n \in \mathbb{N}$, there exist $t_n, T_n \in \mathbb{T}_n$ such that $t_n \le f' \le T_n$ and $\|T_n - f'\| \le 3\tau\ (f',\, 2\pi/n)_p$ and $\|f' - t_n\| \le 3\tau\ (f',\, 2\pi/n)_p$. From Prop. 10.3.1 there exists polynomials $r_n, R_n \in \mathbb{T}_n$ such that

$$r_n(x) \ \le\ f(x) \ \le\ R_n(x)$$

$$\|R_n - f\|_p \ \le\ \left(\frac{\pi}{n}\right)\ \|f' - T_n\|_p \ \le\ \left(\frac{\pi}{n}\right)\ 3\tau\ (f',\, 2\pi/n)_p$$

and

$$\|r_n - f\|_p \ \le\ \left(\frac{\pi}{n}\right)\ \|f' - t_n\|_p \ \le\ \left(\frac{\pi}{n}\right)\ 3\tau\ (f',\, 2\pi/n)_p \,.\, \square$$

10.4 References

[1] A. Andreev, V. A. Popov and B. Sendov, Jackson-type theorems for best one-side approximations by trigonometrical polynomials and splines, Math. Notes, 26 (1979), 889-896.

[2] V. F. Babenko and A. A. Ligun, The order of the best one-sided approximations by polynomials and splines in the metric L_p, Math. Notes, 19 (1976), 194-198.

[3] J. Bustamante, One-sided trigonometric approximation for unbounded functions, to appear.

[4] E. P. Dolzenko and E. A. Sevastyanov, Approximation to functions in the Hausdorff metric using pointwise monotonic (in particular, rational) functions, Mat. Sb., 101 (1976), 508-541.

[5] T. Ganelius, On one-sided approximation by trigonometric polynomials, Math. Scand., 4 (1956), 247-258.

[6] V. I. Ivanov, On one-sided approximations of functions in the L_p-metrics, Sov. Math., Dokl. 18 (1977), 135-138.

[7] W. Jianli and Z. Songping, Some converse results on onesided approximation: justifications, Analysis in Theory and Applications, 19 (3) (2003), 280-288.

[8] V. A. Popov, Converse theorems for one-sided trigonometrical approximation, Compt. Rend. Acad. Bulg. Sci., 30 (1977), 1529-1532.

[9] V. A. Popov and A. S. Andreev, Stechkin's type theorems for one-sided trigonometrical and spline approximation, Compt. Rend. Acad. Bulg. Sci., 31 (1978), 151-154.

[10] A. Yu. Shadrin, Orders of one-sided approximations of functions in L_p-metric, Analysis Mathematica, 12 (1986), 175-184.

Tópicos de Teoría de la Aproximación IV de Jorge Bustamante González, Slaviša V. Djordjević y Miguel A. Jiménez Pozo se terminó de imprimir en octubre de 2012 en los talleres de Siena Editores, con domicilio en Jade 4325 de la colonia Villa Posadas de Puebla, Pue., y con números de teléfono y fax 222 756 8220.

La composición, el diseño y el cuidado de edición son de los autores y la coordinación de la DFE.

El tiraje consta de 300 ejemplares.

9 786074 874679

Benemérita Universidad Autónoma de Puebla
Vicerrectoría de Investigación y Estudios de Posgrado
Facultad de Ciencias Físico Matemáticas
Dirección de Fomento Editorial