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Jorge Bustamante González
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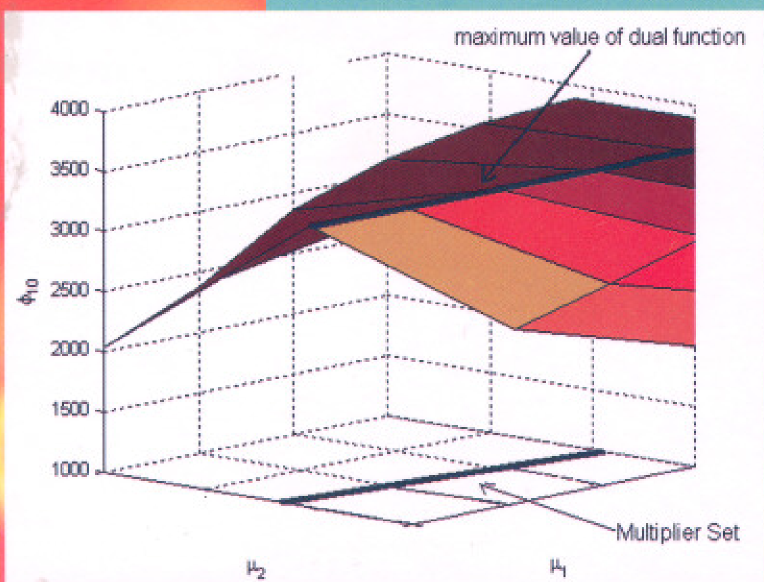
TÓPICOS DE TEORÍA DE LA APROXIMACIÓN III



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Benemérita Universidad Autónoma de Puebla

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Preface

The 3th International Colloquium on Approximation Theory and Related Topics of the University of Puebla was held in May 2008. As its predecessors, though not excluded a priori, it is not the objective of these memories to request new research papers but original works or surveys that present the state of the art in particular subjects in which the authors are active. This kind of monographs are useful for our graduate students as well as for researchers in other areas. With this aim in hands we collect here several talks of the Colloquium and we emphasize that all papers in this volume, the ones of the editors included, have been submitted to usual referee procedure.

I would like to thank the University of Puebla, the Organizing Committee of the Colloquium, their participants, the authors, the anonymous referees and very special to my friends and also editors of this volume Prof. Dr. Jorge Bustamante and Prof. Dr. Slavisa Djordjevic, for their help in the organization and further preparation of this volume. On behalf of the Approximation people of the University of Puebla, Mexico, I am very grateful to all of them for their indispensable support .

Prof. Dr., Dr. Sc. Miguel A. Jiménez-Pozo

Conference Chairman

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Orthogonal polynomials and potential theory. A brief introduction

Guillermo López Lagomasino[1]

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Abstract: We show the connection between Padé approximation and orthogonal polynomials. In particular, how the logarithmic asymptotic of sequences of orthogonal polynomials allows to give the exact rate of convergence of diagonal Padé approximants associated with Markov type functions. We also present the relation between mixed type Hermite-Padé approximants and multiple orthogonal polynomials and state some recent results in this direction.

Keywords and phrases. Orthogonal polynomals, Padé approximation, Nikishin system, logarithmic asymptotic, potential theory.

A.M.S. Subject Classification. Primary: 30E10, 42C05; Secondary: 41A20.

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2.1 Padé approximation and orthogonal polynomials.

Let s be a finite positive Borel measure with compact support $\mathrm{supp}(s)$ contained in the real line consisting of infinitely many points. By $\mathrm{Co}(\mathrm{supp}(s))$ we denote the smallest interval containing $\mathrm{supp}(s)$. Let

$$\widehat{s}(z) = \int \frac{ds(x)}{z-x}$$

be the Cauchy transform of s. Obviously, $\widehat{s} \in \mathcal{H}(\mathbb{C} \setminus \mathrm{supp}(s))$; that is, it is analytic in the complement of the support of the measure.

Lemma 2.1.1 *For each $n \in \mathbb{Z}_+$ there exist polynomials Q_n, P_n such that:*

i) $\deg Q_n \leq n, \deg P_n \leq n-1, Q_n \not\equiv 0,$

ii) $(Q_n \widehat{s} - P_n)(z) = \mathcal{O}(1/z^{n+1}), z \to \infty.$

The rational function $\pi_n = P_n/Q_n$ is uniquely determined by $i) - ii)$.

Proof. Finding Q_n reduces to solving a homogeneous linear system of n equations on $n+1$ unknowns (the coefficients of Q_n). The equations are obtained taking the coefficients of Q_n so that the Laurent expansion of $Q_n \widehat{s}$ at infinity is of the form

$$(Q_n \widehat{s})(z) = P(z) + \mathcal{O}\left(\frac{1}{z^{n+1}}\right), \qquad z \to \infty.$$

That is, so that the Laurent coefficients corresponding to $1/z, \ldots, 1/z^n$ are all equal to zero. Naturally, we then take $P_n = P$.

Suppose that the pairs (Q_n, P_n) and (Q_n^*, P_n^*) both solve i)-ii). Then

$$Q_n^*(z)(Q_n \widehat{s} - P_n)(z) = \mathcal{O}(1/z), \qquad z \to \infty$$

and

$$Q_n(z)(Q_n^* \widehat{s} - P_n^*)(z) = \mathcal{O}(1/z), \qquad z \to \infty.$$

Deleting one of these equations from the other, we get

$$Q_n^*(z)P_n(z) - Q_n(z)P_n^*(z) = \mathcal{O}(1/z), \qquad z \to \infty.$$

Since $Q_n^* P_n - Q_n P_n^*$ is a polynomial, this implies that

$$Q_n^*(z)P_n(z) - Q_n(z)P_n^*(z) \equiv 0.$$

In other words,

$$\pi_n(z) = \frac{P_n(z)}{Q_n(z)} \equiv \frac{P_n^*(z)}{Q_n^*(z)},$$

which is what remained to be proved. $\square$

Definition 2.1.1 *The rational function π_n is called the n-th diagonal Padé approximant associated to $\widehat{s}$.*

In the sequel, we take Q_n with leading coefficient equal to 1; that is, monic.

Lemma 2.1.2 *We have*

$$\int x^\nu Q_n(x) ds(x) = 0, \qquad \nu = 0, \dots, n-1. \tag{2.1}$$

All the zeros of Q_n are simple and lie in the interior of $\text{Co}(\text{supp}(s))$ (*with the euclidean topology of* $\mathbb{R}$). *Additionally,*

$$P_n(z) = \int \frac{Q_n(z) - Q_n(x)}{z - x} ds(x), \tag{2.2}$$

$$(\widehat{s} - \pi_n)(z) = \int \frac{Q(x)Q_n(x)}{Q(z)Q_n(z)} \frac{ds(x)}{z - x}, \tag{2.3}$$

where Q denotes any polynomial of degree $\leq n$.

Proof. Notice that

$$z^\nu (Q_n \widehat{s} - P_n)(z) = \mathcal{O}(1/z^2), \qquad z \to \infty, \qquad \nu = 0, \dots, n-1.$$

Let Γ be a close simple Jordan curve that surrounds $\text{Co}(\text{supp}(s))$ in the positive direction; that is has winding number equal to 1. From Cauchy's theorem, Cauchy's integral formula, and Fubini's theorem, it follows that

$$0 = \frac{1}{2\pi i} \int_\Gamma z^\nu (Q_n \widehat{s} - P_n)(z) dz = \frac{1}{2\pi i} \int_\Gamma z^\nu (Q_n \widehat{s})(z) dz =$$

$$\frac{1}{2\pi i} \int_\Gamma z^\nu Q_n(z) \int \frac{ds(x)}{z - x} dz = \int \frac{1}{2\pi i} \int_\Gamma \frac{z^\nu Q_n(z)}{z - x} dz ds(x) =$$

$$\int x^\nu Q_n(x) ds(x)$$

which is (2.1).

Suppose that Q_n changes sign $N \leq n-1$ times in the interior of Co(supp(s)). Take a polynomial $p, \deg p = N$, with a simple zero at each point where Q_n changes sign in the interior of Co(supp(s)). From (2.1), it follows that

$$\int p(x)Q_n(x)ds(x) = 0.$$

This is impossible because $p\,Q_n$ is a polynomial with a finite number of zeros, with constant sign on supp(s), and this set contains infinitely many points.

Let Q be any polynomial of degree $\leq n$. Notice that

$$Q(z)(Q_n\widehat{s} - P_n)(z) = \mathcal{O}(1/z), \qquad z \to \infty.$$

Take Γ as above with z lying in the unbounded connected component of the complement of Γ. From Cauchy's integral formula, Cauchy's theorem, and Fubini's theorem it follows that

$$Q(z)(Q_n\widehat{s} - P_n)(z) = \frac{1}{2\pi i}\int_\Gamma \frac{Q(\zeta)(Q_n\widehat{s} - P_n)(\zeta)}{z-\zeta}d\zeta$$

$$= \frac{1}{2\pi i}\int_\Gamma \frac{Q(\zeta)(Q_n\widehat{s})(\zeta)}{z-\zeta}d\zeta = \frac{1}{2\pi i}\int_\Gamma \frac{Q(\zeta)Q_n(\zeta)}{z-\zeta}\int \frac{ds(x)}{\zeta - x}dz$$

$$= \int \frac{1}{2\pi i}\int_\Gamma \frac{Q(\zeta)Q_n(\zeta)}{(z-\zeta)(\zeta-x)}d\zeta ds(x) = \int \frac{Q(x)Q_n(x)}{z-x}ds(x).$$

Dividing this equation by $Q(z)Q_n(z)$ we obtain (2.3). Taking $Q \equiv 1$ that formula may be rewritten as

$$P_n(z) = Q_n(z)\widehat{s}(z) - \int \frac{Q_n(x)}{z-x}ds(x) = Q_n(z)\int \frac{ds(x)}{z-x} - \int \frac{Q_n(x)}{z-x}ds(x)$$

which is the same as (2.2). □

Lemma 2.1.3 *Let* $Q_n(z) = \prod_{k=1}^n (z - x_{n,k})$. *Then, for any polynomial* $p, \deg p \leq 2n-1$

$$\int p(x)ds(x) = \sum_{k=1}^n \lambda_{n,k} p(x_{n,k}), \tag{2.4}$$

where

$$\lambda_{n,k} = \int \left(\frac{Q_n(x)}{Q_n'(x_{n,k})(x-x_{n,k})}\right)^j ds(x) > 0, \qquad j = 1,2. \tag{2.5}$$

Moreover,

$$\pi_n(z) = \sum_{k=1}^n \frac{\lambda_{n,k}}{x - x_{n,k}}. \tag{2.6}$$

Proof. Choose $p, \deg p \leq 2n-1$. Let ℓ_{n-1} be the Lagrange polynomial of degree $n-1$ which interpolates p at the n zeros of Q_n. We have

$$(p-\ell_{n-1})(z) = (qQ_n)(z)$$

where q is some polynomial of degree $n-1$. Integrating this equality with respect to the measure s and using orthogonality it follows that

$$0 = \int (p-\ell_{n-1})(x)ds(x).$$

Since

$$\ell_{n-1}(z) = \sum_{k=1}^{n} \frac{Q_n(x)p(x_{n,k})}{Q_n'(x_{n,k})(x-x_{n,k})},$$

substituting this expression in the integral above, we obtain

$$\int p(x)ds(x) = \sum_{k=1}^{n} p(x_{n,k}) \int \frac{Q_n(x)ds(x)}{Q_n'(x_{n,k})(x-x_{n,k})}$$

which is (2.4) with

$$\lambda_{n,k} = \int \frac{Q_n(x)}{Q_n'(x_{n,k})(x-x_{n,k})} ds(x).$$

Notice that

$$\deg \left(\frac{Q_n(x)}{Q_n'(x_{n,k})(x-x_{n,k})} \right)^2 = 2n-2.$$

Substituting this polynomial in (2.4), we obtain (2.5).

Let us decompose π_n into simple fractions

$$\pi_n(z) = \sum_{k=0}^{n} \frac{\beta_{n,k}}{z-x_{n,k}}.$$

Using (2.2), the formula for the residue of a function at a simple pole, and the Lebesgue dominated convergence theorem, we obtain that

$$\beta_{n,k} = \lim_{z\to x_{n,k}} (z-x_{n,k})\pi_n(z) = \lim_{z\to x_{n,k}} \frac{(z-x_{n,k})}{Q_n(z)} \int \frac{Q_n(z)-Q_n(x)}{z-x} ds(x)$$

$$= \int \frac{Q_n(x)}{Q_n'(x_{n,k})(x-x_{n,k})} ds(x) = \lambda_{n,k}.$$

The proof is complete. □

We are ready to prove the well known Markov Theorem which expresses that for any finite positive Borel measure supported on a compact subset of the real line, the sequence of diagonal Padé approximants converges to the corresponding Markov function. Our formulation also contains an estimate of the rate of convergence.

Theorem 2.1.1 *We have*

$$\limsup_n \|\widehat{s} - \pi_n\|_K^{1/2n} \leq \|\varphi\|_K \ (< 1), \tag{2.7}$$

where φ *denotes the conformal representation of* $\overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(s))$ *onto* $\{w : |w| < 1\}$ *such that* $\varphi(\infty) = 0, \varphi'(\infty) > 0$ *and* $\| \cdot \|_K$ *denotes the* sup *norm on the compact set* K *contained in* $\overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(s))$.

Proof. Using (2.6), (2.5), and (2.4) (with $p \equiv 1$), it follows that

$$|\pi_n(z)| \leq \sum_{k=0}^{n} \frac{\lambda_{n,k}}{|z - x_{n,k}|} \leq \frac{|\sigma|}{d_1(z)},$$

where $d_1(z) = \inf\{|z - x| : x \in \mathrm{Co}(\mathrm{supp}(s))\}$. Fix an arbitrary compact subset $K \subset \overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(s))$. From the previous bound, we have that the family of functions $\{f - \pi_n\}, n \in \mathbb{Z}_+$ is uniformly bounded on K.

Fix $\varepsilon, 0 < \varepsilon < 1$, sufficiently close to one so that the compact set K lies in the unbounded connected component of the complement of the level curve $\gamma_\varepsilon = \{z \in \mathbb{C} : |\varphi(z)| = \varepsilon\}$. We have that

$$\sup_{z \in \gamma_\varepsilon} \frac{|(\widehat{s} - \pi_n)(z)|}{|\varphi^{2n+1}(z)|} \leq \frac{1}{\varepsilon^{2n+1}} (\|\widehat{s}\|_{\gamma_\varepsilon} + \|\pi_n\|_{\gamma_\varepsilon}) \leq \frac{C_1(\gamma_\varepsilon)}{\varepsilon^{2n+1}},$$

where $C_1(\gamma_\varepsilon)$ is a positive constant that does not depend on n (notice that γ_ε is also a compact subset of $\overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(s))$. From the definition of π_n (see ii) in Lemma 2.1.1) and the fact that Q_n is of degree n with all its zeros in $\mathrm{Co}(\mathrm{supp}(s))$, we have that the function $(f - \pi_n)/\varphi^{2n+1}$ is analytic in $\overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(s))$ (φ has a simple pole at ∞). Applying the maximum module principle, we obtain that

$$\sup_{z \in K} \frac{|(\widehat{s} - \pi_n)(z)|}{|\varphi^{2n+1}(z)|} \leq \frac{C_1(\gamma_\varepsilon)}{\varepsilon^{2n+1}},$$

which implies that

$$\|\widehat{s} - \pi_n\|_K \leq \frac{C_1(\gamma_\varepsilon)}{\varepsilon^{2n+1}} \|\varphi\|_K.$$

In order to deduce (2.7) it only rests to take $2n$-th root on both sides, then $\limsup$ on n and finally make ε tend to 1. That $\|\varphi\|_K < 1$ is an immediate consequence of the fact that $\varphi(K)$ is a compact subset of the open unit circle. $\square$

That
$$\lim_{n\to\infty} \pi_n(z) = \widehat{s}(z), \qquad K \subset \overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(s)),$$
follows from (2.7) since $\|\varphi\|_K < 1$. This notation stands for uniform convergence on each compact subset K of the indicated region.

Theorem 2.2.2 deals with obtaining the n-th root asymptotic of orthogonal polynomials for special classes of measures. This allows to give the exact limit in (2.7) due to the fact that
$$|(\widehat{s} - \pi_n)(z)| \approx \frac{1}{|q_n^2(z)|},$$
where $q_n = Q_n/\|Q_n\|_2$ and $\|Q_n\|_2^2 = \int |Q_n(x)|^2 ds(x)$. In fact, according to (2.3)
$$(\widehat{s} - \pi_n)(z) = \int \frac{Q_n^2(x)}{Q_n^2(z)} \frac{ds(x)}{z-x} = \int \frac{q_n^2(x)}{q_n^2(z)} \frac{ds(x)}{z-x}.$$
On the other hand, it is easy to verify that
$$0 < \frac{d_1(z)}{d_2^2(z)} \leq \left| \int \frac{q_n^2(x) ds(x)}{z-x} \right| \leq \frac{1}{d_1(z)} < \infty, \qquad z \in \mathbb{C} \setminus \mathrm{Co}(\mathrm{supp}(s)). \tag{2.8}$$
where $d_1(z) = \inf\{|z-x| : x \in \mathrm{Co}(\mathrm{supp}(s))\}, d_2(z) = \max\{|z-x| : x \in \mathrm{Co}(\mathrm{supp}(s))\}$.

The n-th root asymptotic of orthogonal polynomials is obtained using methods from potential theory. This is the aim of the next section.

2.2 Orthogonal polynomials and the logarithmic potential.

Let K be a compact subset of the complex plane $\mathbb{C}$. By $\mathcal{M}(K)$ we denote the set of all finite (positive) Borel measures μ whose support is contained in K.

Definition 2.2.1 *Let $\mu \in \mathcal{M}(K)$, we call logarithmic potential of μ to*
$$p(\mu; z) = \int_K \log \frac{1}{|z-\zeta|} d\mu(\zeta)\,.$$

Notice that for any compact set $E \subset \mathbb{C}$,
$$\log \frac{1}{|z-\zeta|} \geq \log \frac{1}{\mathrm{diam}(E \cup K)}, \quad z \in E, \quad \zeta \in K,$$

where

$$\operatorname{diam}(E \cup K) = \sup\{|z-\zeta| : z \in E, \quad \zeta \in K\}.$$

Therefore,

$$\inf_{z\in E} p(\mu; z) \geq -\mu(K)\log(\operatorname{diam}(E \cup K)) > -\infty.$$

Since the logarithmic kernel is bounded from below on any compact subset of $\mathbb{C}$, one can use of all the Lebesgue integration theory (as if we were dealing with positive functions).

Definition 2.2.2 *The energy associated with* $\mu \in \mathcal{M}(K)$ *is*

$$I_\mu = \iint \log\frac{1}{|z-\zeta|} d\mu(\zeta) d\mu(z) = \int p(\mu; z) d\mu(z)\,.$$

(*The second equality is due to Fubini's theorem.*)

Let E be a compact subset of $\mathbb{C}$. Consider the set $\mathcal{M}_1(E)$ of all probability measures supported on E. Let,

$$I(E) = \inf\{I_\mu : \mu \in \mathcal{M}_1(E)\}.$$

Since $p(\mu; z) \geq -\log(\operatorname{diam}(E))\mu(E)$, it follows that $I(E) > -\infty$. The value $I(E)$ is called *Robin's constant* of E and

$$\mathrm{C}(E) = \exp\{-I(E)\}$$

is called the (logarithmic) *capacity* of E.

Definition 2.2.3 *Let A be an arbitrary subset of the complex plane. The (interior) logarithmic capacity of A is*

$$\mathrm{C}(A) = \sup_E\{\mathrm{C}(E) : E \subset A\}, \quad E \text{ compact}.$$

A direct consequence of the definition is that $A_1 \subset A_2$ implies $\mathrm{C}(A_1) \leq \mathrm{C}(A_2)$.

The following result is an extension of Frostman's theorem for the logarithmic potential. For a proof we refer to [7, Theorem I.1.3].

Theorem 2.2.1 *Let K be a compact subset of the complex plane such that $\mathrm{C}(K) > 0$. Let ϕ be a real valued continuous function on K. Then, there exists a unique $\overline{\nu} \in \mathcal{M}_1(K)$ and a Borel set $e, \mathrm{C}(e) = 0$, such that*

$$p(\overline{\nu}; z) + \phi(z) \begin{cases} \geq w, & z \in K \setminus e, \\ \leq w, & z \in \operatorname{supp}(\overline{\mu}). \end{cases} \tag{2.9}$$

$\overline{\nu}$ and w are called the **equilibrium measure** and the **equilibrium constant**, respectively, which solve the extremal problem (2.9) in the presence of the external field ϕ. When $K \subset \mathbb{R}$ is regular with respect to the Dirichlet problem (that is, the Green function of the region $\overline{\mathbb{C}} \setminus \mathrm{Co}(K)$ can be extended continuously to all $\mathbb{C}$), then $p(\overline{\nu}; z) + \phi(z) \geq w, z \in K$. This is the situation in which we will work.

Definition 2.2.4 *Let s be a finite positive Borel measure supported on a compact set of the real line and let $\{Q_n\}, n \in \mathbb{Z}_+$, be the corresponding sequence of monic orthogonal polynomials. We say that s is a regular measure and write $s \in \mathbf{Reg}$ if*

$$\lim_n \|Q_n\|_2^{1/n} = \mathrm{C}(\mathrm{supp}(s)).$$

For more on the concept of regular measure see [9, Chapter 3], in particular, Theorems 3.1.1 and 3.1.4 therein.

Given an arbitrary polynomial p_l of degree l, the associated normalized zero counting measure is defined to be

$$\nu_{p_l} = \frac{1}{l} \sum_{\{x: p_l(x)=0\}} \delta_x,$$

where δ_x denotes Dirac's delta with mass 1 at x (zeros are repeated in the sum according to their multiplicity).

Given a sequence of measures μ_n and a measure μ supported on a compact set K, we write

$$* \lim_n \mu_n = \mu$$

to mean that for every continuous function f on K

$$\lim_n \int f(x) d\mu_n(x) = \int f(x) d\mu(x).$$

Theorem 2.2.2 *Let $\sigma \in \mathbf{Reg}, \mathrm{supp}(\sigma) \subset \mathbb{R}$, where $\mathrm{supp}(\sigma)$ is regular with respect to the Dirichlet problem. Let $\{\phi_l\}, l \in \Lambda \subset \mathbb{Z}_+$, be a sequence of positive continuous functions on $\mathrm{supp}(\sigma)$ such that*

$$\lim_{l \in \Lambda} \frac{1}{2l} \log \frac{1}{|\phi_l(x)|} = \phi(x) > -\infty, \tag{2.10}$$

uniformly on $\mathrm{supp}(\sigma)$. By $\{q_l\}, l \in \Lambda$, denote a sequence of monic polynomials such that $\deg q_l = l$ and

$$\int x^k q_l(x) \phi_l(x) d\sigma(x) = 0, \qquad k = 0, \ldots, l-1. \tag{2.11}$$

Then

$$*\lim_{l\in\Lambda} \nu_{q_l} = \overline{\nu}, \tag{2.12}$$

$$\lim_{l\in\Lambda}\left(\int |q_l(x)|^2\phi_l(x)d\sigma(x)\right)^{1/2l} = e^{-w}, \tag{2.13}$$

and

$$\lim_{l\in\Lambda}\left(\frac{|q_l(z)|}{\|q_l\phi_l^{1/2}\|_E}\right)^{1/l} = \exp\left(w - V^{\overline{\nu}}(z)\right), \quad K \subset \mathbb{C}\setminus \mathrm{Co}(\mathrm{supp}(\sigma)). \tag{2.14}$$

where $\overline{\nu}$ *and* w *are the unique equilibrium measure and equilibrium constant in the presence of the external field* ϕ *on* $\mathrm{supp}(\sigma)$*.*

Proof. On account of (2.10) and Theorem 2.2.1, it follows that for any $\varepsilon > 0$ there exists l_0 such that for all $l \geq l_0, l \in \Lambda$, and $z \in \mathrm{supp}(\overline{\nu}) \subset \mathrm{supp}(\sigma) := E$

$$\frac{1}{l}\log\frac{|p_l(z)|}{\|p_l\phi_l^{1/2}\|_E} \leq \frac{1}{2l}\log\frac{1}{|\phi_l(z)|} \leq \phi(z) + \varepsilon \leq w - V^{\overline{\nu}}(z) + \varepsilon,$$

where $\{p_l\}, l \in \Lambda$, is any sequence of monic polynomials such that $\deg p_l = l$ and $\|p_l\phi_l^{1/2}\|_E = \max_{z\in E}|(p_l\phi_l^{1/2})(z)|$. Hence,

$$u_l(z) := V^{\overline{\nu}}(z) + \frac{1}{l}\log\frac{|p_l(z)|}{\|p_l\phi_l^{1/2}\|_E} \leq w + \varepsilon, \quad z \in E, \quad l \geq l_0.$$

Since u_l is subharmonic in $\overline{\mathbb{C}}\setminus\mathrm{supp}(\overline{\nu})$, by the maximum principle, we have

$$u_l(z) \leq w + \varepsilon, \quad z \in \overline{\mathbb{C}}, \quad l \geq l_0.$$

In particular,

$$u_l(\infty) = \frac{1}{l}\log\frac{1}{\|p_l\phi_l^{1/2}\|_E} \leq w + \varepsilon.$$

These two relations imply

$$\limsup_{l\in\Lambda}\left(\frac{|p_l(z)|}{\|p_l\phi_l^{1/2}\|_E}\right)^{1/l} \leq \exp\left(w - V^{\overline{\nu}}(z)\right), \qquad K \subset \mathbb{C}, \tag{2.15}$$

and

$$\liminf_{l\in\Lambda}\|p_l\phi_l^{1/2}\|_E^{1/l} \geq \exp(-w). \tag{2.16}$$

In particular, the sequence of polynomials $\{q_l\}, l \in \Lambda$, of the lemma satisfies these relations.

Let t_l be the weighted Fekete polynomial of degree l associated with the weight $e^{-\phi}$ on $\operatorname{supp}(\sigma)$ and $|\sigma|$ be the total variation of σ. From the minimality property in the L_2 norm of q_l, we have

$$\|q_l\phi_l^{1/2}\|_2 := \left(\int |q_l(x)|^2\phi_l(x)d\sigma(x)\right)^{1/2} \|t_l\phi_l^{1/2}\|_2 \leq |\sigma|^{1/2}\|t_l\phi_l^{1/2}\|_E \leq$$

$$\leq |\sigma|^{1/2}\|t_l e^{-l\phi}\|_E\|\phi_l^{1/2}e^{l\phi}\|_E.$$

Then, using (2.10) and Theorem III.1.9 in [7], we obtain that

$$\limsup_{l\in\Lambda} \|q_l\phi_l^{1/2}\|_2^{1/l} \leq e^{-w}. \tag{2.17}$$

Since $\operatorname{supp}(\sigma)$ is regular with respect to the Dirichlet problem, Theorem 3.2.3 vi) in [7] yieds

$$\lim_{l\in\Lambda}\left(\frac{\|q_l\phi_l^{1/2}\|_E}{\|q_l\phi_l^{1/2}\|_2}\right)^{1/l} = 1. \tag{2.18}$$

The relations (2.16), (2.17), and (2.18) imply

$$\limsup_{l\in\Lambda} \|q_l\phi_l^{1/2}\|_E^{1/l} = \limsup_{l\in\Lambda} \|q_l\phi_l^{1/2}\|_2^{1/l} = e^{-w}. \tag{2.19}$$

In particular, we obtain (2.13).

All the zeros of q_l lie in $\operatorname{Co}(\operatorname{supp}(\sigma)) \subset \mathbb{R}$. The unit ball in the weak topology of measures is relatively compact. Take any subsequence of indices $\Lambda' \subset \Lambda$ such that

$$*\lim_{l\in\Lambda'} \nu_{q_l} = \nu_{\Lambda'}.$$

Then,

$$\lim_{l\in\Lambda'} \frac{1}{l}\log|q_l(z)| = -\lim_{n\in\Lambda'}\int \log\frac{1}{|z-x|}\nu_{q_l}(x) = -V^{\nu_{\Lambda'}}(z),$$

$\mathcal{K} \subset \mathbb{C}\setminus\operatorname{Co}(\operatorname{supp}(\sigma))$. This, together with (2.13) and (2.15) imply

$$(V^{\overline{\nu}} - V^{\nu_{\Lambda'}})(z) \leq 0, \qquad z \in \overline{\mathbb{C}}\setminus\operatorname{Co}(\operatorname{supp}(\sigma)).$$

Since $V^{\overline{\nu}} - V^{\nu_{\Lambda'}}$ is subharmonic in $\overline{\mathbb{C}}\setminus(\operatorname{supp}(\overline{\nu}))$ and $(V^{\overline{\nu}} - V^{\nu_{\Lambda'}})(\infty) = 0$, from the maximum principle, it follows that $V^{\overline{\nu}} \equiv V^{\nu_{\Lambda'}}$ in $\mathbb{C}\setminus\operatorname{Co}(\operatorname{supp}(\sigma))$ and thus $\nu_{\Lambda'} = \overline{\nu}$. Consequently, (2.12) holds.

Finally, (2.14) is a direct consequence of (2.12) and (2.13). □

From Theorem 2.2.2, (2.3), and (2.8), we obtain:

Theorem 2.2.3 *Let $\sigma \in \mathbf{Reg}, \mathrm{supp}(\sigma) \subset \mathbb{R}$, where $\mathrm{supp}(\sigma)$ is regular with respect to the Dirichlet problem. Then*

$$\lim_n \|\widehat{s} - \pi_n\|_K^{1/2n} = \|e^{p(\overline{\nu};\cdot)-w}\|_K,$$

on each compact subset $K \subset \mathbb{C} \setminus \mathrm{Co}(\mathrm{supp}(s))$.

2.3 Mixed type Hermite-Padé approximation and multiple orthogonal polynomials

Let $\mathbb{F} = (F_{k,j}), 0 \leq k \leq m_2, 0 \leq j \leq m_1$, be a matrix of functions, with $m_2 + 1$ rows and $m_1 + 1$ columns, all of which are analytic in some region Ω of the complex plane containing ∞. Fix $\mathbf{n}_1 = (n_{1,0}, n_{1,1}, \ldots, n_{1,m_1}) \in \mathbb{Z}_+^{m_1+1}$ and $\mathbf{n}_2 = (n_{2,0}, n_{2,1}, \ldots, n_{2,m_2}) \in \mathbb{Z}_+^{m_2+1}$. Set $|\mathbf{n}_1| = n_{1,0} + n_{1,1} + \cdots + n_{1,m_1}$, $|\mathbf{n}_2| = n_{2,0} + \cdots + n_{2,m_2}$, and $\mathbf{n} = (\mathbf{n}_1; \mathbf{n}_2)$. In the sequel, we suppose that $|\mathbf{n}_2| + 1 = |\mathbf{n}_1|$. We wish to find polynomials $a_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m_1}$, such that

i) $\deg(a_{\mathbf{n},j}) \leq n_{1,j} - 1$, $j = 0, \ldots, m_1$, not all identically equal to zero.

ii) For $k = 0, \ldots, m_2$

$$\sum_{j=0}^{m_1} a_{\mathbf{n},j}(z) F_{k,j}(z) - d_{\mathbf{n},k}(z) = \mathcal{O}\left(1/z^{n_{2,k}+1}\right), \qquad z \to \infty, \tag{2.20}$$

where $d_{\mathbf{n},k}$ is also a polynomial ($\deg(a_{\mathbf{n},j}) \leq -1$ means that $a_{\mathbf{n},j} \equiv 0$).

Finding $a_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m}$ reduces to solving a homogeneous linear system of $|\mathbf{n}_2|$ equations on $|\mathbf{n}_1|$ unknowns (the coefficients of the $a_{\mathbf{n},j}$). The equations are obtained equating to zero the coefficients of $1/z, \ldots, 1/z^{n_{2,k}}$ of the Laurent expansion at ∞ of the function $\sum_{j=0}^{m_1} a_{\mathbf{n},j}(z) F_{k,j}(z)$. Then $d_{\mathbf{n},k}$ is taken as the polynomial part of this Laurent expansion. Since $|\mathbf{n}_2| = |\mathbf{n}_1| - 1$ a non-trivial solution exists.

When $m_2 = 0$, this construction is called type I Hermite-Padé approximation. If $m_1 = 0$ it is called of type II. When $m_1 = m_2 = 0$ it reduces to diagonal Padé approximation. The construction presented here is of mixed type. Type I Hermite-Padé approximation is used in questions related with the algebraic independence of functions and numbers. Those of type II are used in simultaneous approximation of vector functions by means of vector rational functions with a common denominator. A combination of the two types appear in stochastic models connected with random matrices and non intersecting random paths (see [2]). The algebraic properties of mixed type Hermite Padé approximants were studied in [8].

Let us restrict the class of matrix functions under consideration in order to relate this construction with so called multiple orthogonal polynomials.

Let s be a finite positive Borel measure with compact support $\mathrm{supp}(s)$ contained in the real line consisting of infinitely many points. Let $(w_{k,j}), 0 \leq k \leq m_2, 0 \leq j \leq m_1$, be a matrix of integrable functions with respect to s. For each k, j assume that

$$F_{k,j}(z) = \int \frac{w_{k,j}(x)ds(x)}{z-x}.$$

For the matrix of functions $\mathbb{F}$ associated with these functions $F_{k,j}$ find polynomials $a_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m_1}$, that solve i)-ii).

Lemma 2.3.1 *For* $k = 0, \ldots, m_2$

$$\int x^\nu \sum_{j=0}^{m_1} a_{\mathbf{n},j}(x) w_{k,j}(x) ds(x) = 0, \qquad \nu = 0, \ldots, n_{2,k} - 1. \tag{2.21}$$

Proof. In fact, notice that according to ii), for each $\nu, 0 \leq \nu \leq n_{2,k} - 1$,

$$z^\nu \left(\sum_{j=0}^{m_1} a_{\mathbf{n},j}(z) F_{k,j}(z) - d_{\mathbf{n},k}(z) \right) = \mathcal{O}\left(1/z^2\right), \qquad z \to \infty.$$

and the function on the left hand side is holomorphic in $\overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(s))$. Let Γ be a close simple Jordan curve that surrounds $\mathrm{Co}(\mathrm{supp}(s))$ in the positive direction. From Cauchy's theorem, Cauchy's integral formula, and Fubini's theorem, it follows that

$$0 = \frac{1}{2\pi i} \int_\Gamma z^\nu \left(\sum_{j=0}^{m_1} a_{\mathbf{n},j}(z) F_{k,j}(z) - d_{\mathbf{n},k}(z) \right) dz$$

$$= \frac{1}{2\pi i} \int_\Gamma z^\nu \sum_{j=0}^{m_1} a_{\mathbf{n},j}(z) F_{k,j}(z) dz = \frac{1}{2\pi i} \int_\Gamma z^\nu \sum_{j=0}^{m_1} a_{\mathbf{n},j}(z) \int \frac{w_{k,j}(x)ds(x)}{z-x} dz$$

$$= \int \sum_{j=0}^{m_1} \frac{w_{k,j}(x)}{2\pi i} \int_\Gamma \frac{z^\nu a_{\mathbf{n},j}(z)}{z-x} dz ds(x) = \int x^\nu \sum_{j=0}^{m_1} a_{\mathbf{n},j}(x) w_{k,j}(x) ds(x) = 0.$$

$\square$

Now, assume that

$$w_{k,j}(x) = u_k(x) v_j(x). \tag{2.22}$$

Then, we can rewrite (2.21) as

$$\int x^\nu u_k(x) \sum_{j=0}^{m_1} a_{\mathbf{n},j}(x) v_j(x) ds(x) = 0, \qquad \nu = 0,\ldots,n_{2,k}-1. \tag{2.23}$$

Taking linear combinations of (2.23), we obtain that

$$\int \sum_{k=0}^{m_2} b_{\mathbf{n},k}(x) u_k(x) \sum_{j=0}^{m_1} a_{\mathbf{n},j}(x) v_j(x) ds(x) = 0, \tag{2.24}$$

for any system of polynomials $b_{\mathbf{n},k}, \deg b_{\mathbf{n},k} \leq \mathbf{n}_{2,k} - 1, k = 0,\ldots,m_2$.

Let us denote introduce the row vectors

$$\mathbb{U} = (u_0,\ldots,u_{m_2}), \qquad \mathbb{V} = (v_0,\ldots,v_{m_1})$$

$$\mathbb{B}_{\mathbf{n}} = (b_{\mathbf{n},0},\ldots,b_{\mathbf{n},m_2}), \quad \mathbb{A}_{\mathbf{n}} = (a_{\mathbf{n},0},\ldots,a_{\mathbf{n},m_1})$$

and the $m_2 + 1 \times m_1 + 1$ dimensional matrix

$$\mathbb{W} = \mathbb{U}^t \mathbb{V},$$

where the super-index t means taking transpose. Then (2.24) can be written in matrix form as

$$\int \mathbb{B}_{\mathbf{n}}(x) \mathbb{W}(x) \mathbb{A}^t_{\mathbf{n}}(x) ds(x) = 0, \tag{2.25}$$

for any vector polynomial $\mathbb{B}_{\mathbf{n}}$ whose components satisfy $\deg b_{\mathbf{n},k} \leq \mathbf{n}_{2,k} - 1, k = 0,\ldots,m_2$.

Let us introduce the matrix function

$$\widehat{\mathbb{S}}(z) = \int \frac{\mathbb{W}(x) ds(x)}{z - x}$$

understanding that integration is carried out entry by entry on the matrix $\mathbb{W}$. For the matrix function $\widehat{\mathbb{S}}$, the definition given in the beginning of this section can be restated as: there exists a non zero vector polynomial $\mathbb{A}_{\mathbf{n}}, \deg a_{\mathbf{n},j} \leq \mathbf{n}_{1,j} - 1, j = 0,\ldots,m_1$, such that

$$\widehat{\mathbb{S}}(z) \mathbb{A}^t_{\mathbf{n}}(z) - \mathbb{D}^t_{\mathbf{n}}(z) = \mathcal{O}(1/z^{\mathbf{n}_2+1}), \tag{2.26}$$

for some vector polynomial $\mathbb{D}_{\mathbf{n}}$ where the right hand side of (2.26) should be understood as in (2.20). The vector $\mathbb{A}_{\mathbf{n}}$ satisfies the orthogonality relations (2.25) with respect to any vector polynomial $\mathbb{B}_{\mathbf{n}}, \deg b_{\mathbf{n},k} \leq \mathbf{n}_{2,k} - 1, k = 0,\ldots,m_2$. The vector polynomials $\mathbb{A}_{\mathbf{n}}$ are called mixed type multiple orthogonal polynomials.

2.4 Multiple orthogonal polynomials

The purpose of this section is to present some recent developments in the analytic theory of mixed type multiple orthogonal polynomials. We will restrict our attention to mixed type multiple orthogonal polynomials in which the linear forms are generated by two (not necessarily distinct) Nikishin systems of measures. Before going into details let us mention two papers which are of interest in this subject.

E. M. Nikishin discussed the asymptotic behavior of linear forms generated by what are known as Nikishin systems in [5] (see also last section in [6] and [4]). He described the logarithmic asymptotic behavior of type I multiple orthogonal polynomials in terms of the solution of a vector equilibrium problem for the logarithmic potential. Later, Gonchar-Rakhmanov-Sorokin studied in [3] the rate of convergence of Hermite-Padé approximation of generalized Nikishin system of functions and the logarithmic asymptotic of their associated type II multiple orthogonal polynomials. The solution is also characterized by a similar vector equilibrium problem. Our results contain as extreme cases both situations.

Let $S = (s_0^1, \ldots, s_{m_1}^1)$ be a system of finite Borel measures with constant sign, and bounded support consisting of infinitely many points contained in the real line. Let $\widehat{S} = (\widehat{s}_1, \ldots, \widehat{s}_{m_1})$ be the corresponding system of Markov functions; that is,

$$\widehat{s}_k(z) = \int \frac{ds_k(x)}{z-x}, \quad k = 1, \ldots, m_1. \tag{2.27}$$

A special system of Markov functions was introduced by E. M. Nikishin in [5]. They constitute an important model class of functions in the analytic theory of multiple orthogonal polynomials and simultaneous rational approximations since many classical results of these theories have found their corresponding analogues; thus, have attracted increasing attention in recent decades. Let us define them.

Let σ_α, σ_β be two measures with constant sign supported on $\mathbb{R}$ and let Δ_α, Δ_β denote the smallest intervals containing their supports, $\mathrm{supp}(\sigma_\alpha)$ and $\mathrm{supp}(\sigma_\beta)$, respectively. Assume that $\Delta_\alpha \cap \Delta_\beta = \emptyset$ and define

$$\langle \sigma_\alpha, \sigma_\beta \rangle(x) = \int \frac{d\sigma_\beta(t)}{x-t} d\sigma_\alpha(x) = \widehat{\sigma}_\beta(x) d\sigma_\alpha(x).$$

Hence, $\langle \sigma_\alpha, \sigma_\beta \rangle$ is a measure with constant sign and support equal to that of σ_α.

Definition 2.4.1 *For a system of intervals $\Delta_0, \ldots, \Delta_{m_1}$ contained in $\mathbb{R}$ satisfying $\Delta_j \cap \Delta_{j+1} = \emptyset$, $j = 0, \ldots, m_1 - 1$, and finite Borel measures $\sigma_0, \ldots, \sigma_{m_1}$ with constant sign in $\mathrm{Co}(\mathrm{supp}(\sigma_j)) = \Delta_j$, such that each one has infinitely many points in its support, we define recursively*

$$\langle \sigma_0, \sigma_1, \ldots, \sigma_j \rangle = \langle \sigma_0, \langle \sigma_1, \ldots, \sigma_j \rangle \rangle, \quad j = 1, \ldots, m_1.$$

We say that $(s_0, \ldots, s_{m_1}) = \mathcal{N}(\sigma_0, \ldots, \sigma_{m_1})$, *where*

$$s_0 = \langle \sigma_0 \rangle = \sigma_0, \quad s_1 = \langle \sigma_0, \sigma_1 \rangle, \ldots, \quad s_{m_1} = \langle \sigma_0, \ldots, \sigma_{m_1} \rangle$$

is the Nikishin system of measures generated by $(\sigma_0, \ldots, \sigma_{m_1})$.

In the sequel, when referring to a Nikishin system the condition $\Delta_j \cap \Delta_{j+1} = \emptyset, j = 0, \ldots, m-1$, is always assumed to hold. Notice that all the measures in a Nikishin system have the same support, namely $\mathrm{supp}(\sigma_0)$. We will denote

$$s_{j,k} = \langle \sigma_j, \ldots, \sigma_k \rangle, \qquad 0 \le j \le k \le m$$

$(s_{j,j} = \sigma_j)$.

Take two systems

$$S^1 = (s^1_0, \ldots, s^1_{m_1}) = \mathcal{N}(\sigma^1_0, \ldots, \sigma^1_{m_1}), S^2 = (s^2_0, \ldots, s^2_{m_2}) = \mathcal{N}(\sigma^2_0, \ldots, \sigma^2_{m_2})$$

generated by $m_1 + 1$ and $m_2 + 1$ measures respectively. The two systems need not coincide but we will assume that $\sigma^1_0 = \sigma^2_0$; that is, both systems stem from the same basis measure. The smallest interval containing the supports of these measures will be denoted $\mathrm{Co}(\mathrm{supp}(\sigma^i_j)) = \Delta^i_j$.

Fix $\mathbf{n}_1 = (n_{1,0}, n_{1,1}, \ldots, n_{1,m_1}) \in \mathbb{Z}^{m_1+1}_+$ and $\mathbf{n}_2 = (n_{2,0}, n_{2,1}, \ldots, n_{2,m_2}) \in \mathbb{Z}^{m_2+1}_+$. Set $|\mathbf{n}_1| = n_{1,0} + n_{1,1} + \cdots + n_{1,m_1}$, $|\mathbf{n}_2| = n_{2,0} + \cdots + n_{2,m_2}$, and $\mathbf{n} = (\mathbf{n}_1; \mathbf{n}_2)$. As above, we assume that $|\mathbf{n}_2| + 1 = |\mathbf{n}_1|$.

Define

$$\mathbb{W} = \mathbb{U}^t \mathbb{V},$$

where

$$\mathbb{U} = (1, \widehat{s}^2_{1,1}, \ldots, \widehat{s}^2_{1,m_2}), \qquad \mathbb{V} = (1, \widehat{s}^1_{1,1}, \ldots, \widehat{s}^1_{1,m_1}),$$

and take $s = \sigma^1_0 = \sigma^2_0$. With this $\mathbb{W}$ and s, the vector polynomial $\mathbb{A}_{\mathbf{n}}$ that satisfies (2.25) is called the mixed type multiple orthogonal polynomial with respect to the pair of Nikishin systems (S^1, S^2) and the multi-index $\mathbf{n}$. Sometimes it is better to see these orthogonality relations in expanded form. They say that for each $k = 0, \ldots, m_2$,

$$\int x^\nu (a_{\mathbf{n},0}(x) + \sum_{j=1}^{m_1} a_{\mathbf{n},j}(x) \widehat{s}^1_{1,j}(x)) ds^2_{0,k}(x) = 0, \quad \nu = 0, \ldots, n_{2,k} - 1. \tag{2.28}$$

A multi-index $\mathbf{n} = (\mathbf{n}_1; \mathbf{n}_2)$ is said to be **normal** if every non trivial solution $\mathbb{A}_{\mathbf{n}}$ to (2.28) satisfies $\deg a_{\mathbf{n},j} = n_{1,j} - 1, j = 0, \ldots, m$. If $\mathbf{n}$ is normal, it is easy to verify that the vector $\mathbb{A}_{\mathbf{n}}$ is uniquely determined except for a constant factor. Set

$$\mathbb{Z}^{m_1+1}_+(\bullet) = \{\mathbf{n}_1 \in \mathbb{Z}^{m_1+1}_+ : n_{1,0} \ge \cdots \ge n_{1,m_1}\}.$$

It can be proved that all $\mathbf{n} = (\mathbf{n}_1; \mathbf{n}_2) \in \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ are normal. We normalize $\mathbb{A}_{\mathbf{n}}$ so that $a_{\mathbf{n},m_1}$ is monic when $n_{1,m_1} \geq 1$.

Theorem 2.4.1 gives the rate of convergence of the $|\mathbf{n}_1|$-th root of the linear forms

$$\mathcal{A}_{\mathbf{n}}(z) = a_{\mathbf{n},0}(z) + \sum_{k=1}^{m_1} a_{\mathbf{n},k}(z)\widehat{s}^1_{1,k}(z).$$

under mild conditions on the sequence of multi-indices assuming that the measures generating both Nikishin systems belong to the class **Reg** of regular measures. For short, we write $(S^1, S^2) \in \mathbf{Reg}$ to mean that all the measures which generate both Nikishin systems are regular and their supports are regular with respect to the Dirichlet problem. Before stating Theorem 1, we need to introduce some notation and results from potential theory.

Let E_k, $k = -m_2, \ldots, m_1$, be (not necessarily distinct) compact subsets of the real line and $\mathcal{C} = (c_{j,k}), -m_2 \leq j,k \leq m_1$, a real, positive definite, symmetric matrix of order $m_1 + m_2 + 1$. $\mathcal{C}$ will be called the interaction matrix. By $\mathcal{M}(E_k)$ we denote the class of all finite, positive, Borel measures with compact support consisting of an infinite set of points contained in E_k and $\mathcal{M}_1(E_k)$ is the subclass of probability measures of $\mathcal{M}(E_k)$. Set

$$\mathcal{M}_1 = \mathcal{M}_1(E_{-m_2}) \times \cdots \times \mathcal{M}_1(E_{m_1})\,.$$

Given a vector measure

$$\mu = (\mu_{-m_2}, \ldots, \mu_{m_1}) \in \mathcal{M}_1 \quad \text{and} \quad j = -m_2, \ldots, m_1,$$

we define the combined potential

$$W_j^{\mu}(x) = \sum_{k=-m_2}^{m_1} c_{j,k} V^{\mu_k}(x)\,, \tag{2.29}$$

where

$$V^{\mu_k}(x) = \int \log \frac{1}{|x-t|}\, d\mu_k(t)\,,$$

denotes the standard logarithmic potential of μ_k. Set

$$\omega_j^{\mu} = \inf\{W_j^{\mu}(x) : x \in \Delta_j\}\,, \quad j = -m_2, \ldots, m_1\,.$$

In Chapter 5 of [6] the authors prove (we state the result in a form convenient for our purpose).

Lemma 2.4.1 *Let $\mathcal{C}$ be a real, positive definite, symmetric matrix of order $m_1 + m_2 + 1$. If there exists $\overline{\mu} = (\overline{\mu}_{-m_2}, \ldots, \overline{\mu}_{m_1}) \in \mathcal{M}_1$ such that for each $j = -m_2, \ldots, m_1$*

$$W_j^{\overline{\mu}}(x) = \omega_j^{\overline{\mu}}\,, \qquad x \in \operatorname{supp}(\overline{\mu}_j)\,,$$

then $\overline{\mu}$ is unique. Moreover, if $c_{j,k} \geq 0$ when $E_j \cap E_k \neq \emptyset$ and the compact sets E_k are regular with respect to the Dirichlet problem then $\overline{\mu}$ exists.

For details on this lemma see section 4 in [1]. The vector measure $\overline{\mu} \in \mathcal{M}_1$ is called the equilibrium solution for the vector potential problem determined by the interaction matrix $\mathcal{C}$ on the system of compact sets $E_j\,, j = -m_2, \ldots, m_1\,.$

Let $\Lambda = \Lambda(p_{1,0}, \ldots, p_{1,m_1}; p_{2,0}, \ldots, p_{2,m_2}) \subset \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$ be an infinite sequence of distinct multi-indices such that

$$\lim_{\mathbf{n}_1 \in \Lambda} \frac{n_{1,j}}{|\mathbf{n}_1|} = p_{1,j} \in (0,1),\ \ j = 0, \ldots, m_1,$$

$$\lim_{\mathbf{n}_1 \in \Lambda} \frac{n_{2,j}}{|\mathbf{n}_1|} = p_{2,j} \in (0,1),\ \ j = 0, \ldots, m_2.$$

Obviously, $p_{1,0} \geq \cdots \geq p_{1,m_1}, p_{2,0} \geq \cdots \geq p_{2,m_2}$, and

$$\sum_{j=0}^{m_1} p_{1,j} = \sum_{j=0}^{m_2} p_{2,j} = 1.$$

Set

$$P_j = \sum_{k=j}^{m_1} p_{1,k},\ \ j = 0, \ldots, m_1, \qquad P_{-j} = \sum_{k=j}^{m_2} p_{2,k},\ \ j = 0, 1, \ldots, m_2.$$

Let us define the interaction matrix $\mathcal{C}$ which is relevant for the rest of the paper. Take the tri-diagonal matrix

$$\mathcal{C} = \begin{pmatrix} P_{-m_2}^2 & -\frac{P_{-m_2}P_{-m_2+1}}{2} & 0 & \cdots & 0 \\ -\frac{P_{-m_2}P_{-m_2+1}}{2} & P_{-m_2+1}^2 & -\frac{P_{-m_2+1}P_{-m_2+2}}{2} & \cdots & 0 \\ 0 & -\frac{P_{-m_2+1}P_{-m_2+2}}{2} & P_{-m_2+2}^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & P_{m_1}^2 \end{pmatrix}. \tag{2.30}$$

This matrix satisfies all the assumptions of Lemma 2.4.1 with $E_j = \operatorname{supp}(\sigma_j^1), j = 0, 1, \ldots, m_2, E_j = \operatorname{supp}(\sigma_{-j}^2), j = 0, -1, \ldots, -m_2$ (recall that

$\sigma_0^1 = \sigma_0^2$) including $c_{j,k} \geq 0$ when $E_j \cap E_k \neq \emptyset$, and it is positive definite because the principal section $\mathcal{C}_r, r = 1, \ldots, m_1 + m_2 + 1$, of $\mathcal{C}$ satisfies

$$\det(\mathcal{C}_r) = P_{-m_2}^2 \cdots P_{-m_2+r-1}^2 \det \begin{pmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & \cdots & -\frac{1}{2} & 1 \end{pmatrix}_{r \times r} > 0.$$

Let $\overline{\mu}(\mathcal{C})$ be the equilibrium solution for the corresponding vector potential problem.

Theorem 2.4.1 *Let* $(S^1, S^2) \in \mathbf{Reg}$,

$$S^1 = \mathcal{N}(\sigma_0^1, \ldots, \sigma_{m_1}^1), S^2 = \mathcal{N}(\sigma_0^2, \ldots, \sigma_{m_2}^2),$$

and

$$\Lambda = \Lambda(p_{1,0}, \ldots, p_{1,m_1}; p_{2,0}, \ldots, p_{2,m_2}) \subset \mathbb{Z}_+^{m_1+1}(\bullet) \times \mathbb{Z}_+^{m_2+1}(\bullet)$$

be given. Let $\{\mathcal{A}_{\mathbf{n},0}\}, \mathbf{n} \in \Lambda$, *be the sequences of linear forms associated with the corresponding normalized mixed type orthogonal polynomials. Then,*

$$\lim_{\mathbf{n} \in \Lambda} |\mathcal{A}_{\mathbf{n},0}(z)|^{1/|\mathbf{n}_1|} = \mathcal{A}_0(z)\,, \tag{2.31}$$

uniformly on each compact subset of $\mathbb{C} \setminus (\Delta_0^1 \cup \Delta_1^1)$, *where*

$$\mathcal{A}_0(z) = \exp\left(P_1 V^{\overline{\mu}_1}(z) - V^{\overline{\mu}_0}(z) - 2 \sum_{k=1}^{m_1} \frac{\omega_k^{\overline{\mu}}}{P_k} \right).$$

$\overline{\mu} = \overline{\mu}(\mathcal{C}) = (\overline{\mu}_{-m_2}, \ldots, \overline{\mu}_{m_1})$ *is the equilibrium vector measure and*

$$(\omega_{-m_2}^{\overline{\mu}}, \ldots, \omega_{m_1}^{\overline{\mu}})$$

is the system of equilibrium constants for the vector potential problem determined by the interaction matrix $\mathcal{C}$ *defined in* (2.30) *on the system of compact sets* $E_j = \mathrm{supp}(\sigma_j^1), j = 0, \ldots, m_1, E_j = \mathrm{supp}(\sigma_{-j}^2), j = -m_2, \ldots, 0$.

Relation (2.31) is the analogue of (2.14) in this context. This theorem is joint work with Ulises Fidalgo and Abey López and its proof will appear in a forthcoming paper.

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Convexification of the Lagrangian at a strongly stable local minimizer

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Abstract: A local convexification result of the Lagrangian function for a general class of nonconvex optimization problems with inequality constraints was obtained by D. Li and X. L. Sun [4], by using a transformation equivalent to taking pth power of the objective function and the constraints, under the linear independence constraint qualification and the second-order sufficiency condition. In this paper, we show that, the same result can be obtained under a weaker condition, namely, the local minimizer is strongly stable in the sense of M. Kojima [3].

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3.1 Introduction

We consider a nonconvex optimization problem of the following form:

$$P\text{: } \min \ \{f(x) \mid g_j(x) \leq b_j, j \in J, x \in X\},$$

where $f, g_j : \mathbb{R}^n \to \mathbb{R}, j \in J$ $(|J| < \infty)$ are twice continuously differentiable functions and X is a nonempty closed set in $\mathbb{R}^n$. For $h : \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}^n$, $x = (u, v) \in \mathbb{R}^m \times \mathbb{R}^p$, the row vectors $Dh(x)$, $D_u h(x)$ denote the derivative and partial derivative of h with respect to x and u, respectively, and $D_x^2 h(x)$, $D_u^2 h(x)$ denote the Hessian and partial Hessian of h with respect to x and u, respectively. We say that the Linear Independence Constraint Qualification (LICQ) is satisfied at a feasible point $\bar{x}$ of P if $Dg_j(\bar{x}), j \in J_0(\bar{x}) := \{j \in J | g_j(\bar{x}) = b_j\}$ are linearly independent.

In [4], it is proved that, under LICQ and the second-order suffciency condition, the local convexity condition in the local duality theorem can be achieved at a local optimal solution of P after adopting some suitable convexification transformation on problem P.

In this paper we extend the results of [4] to the case of a strongly stable (c.f. [3]) local minimizer of P. We assume that $f(x) > 0, g_j(x) > 0, b_j > 0$, for all $x \in X$ and all $j \in J$. Otherwise, we can apply some suitable equivalent transformation on P if necessary, e.g. exponential transformation.
We assume in the sequel that any local minimizer of P is an interior point of X.

3.2 Theoretical background

Denote

$$M = \{x \in \mathbb{R}^n \mid g_j(x) \leq b_j, \ j \in J, \ x \in X\}.$$

A feasible point $\bar{x} \in M$, $\bar{x} \in \text{int } X$, is called a stationary point for the problem P if there exist numbers $\mu_j \geq 0, j \in J$ such that

$$Df(\bar{x}) + \sum_{j \in J} \mu_j Dg_j(\bar{x}) = 0, \tag{3.1}$$

$$\mu_j \left[g_j\left(\bar{x}\right) - b_j \right] = 0, j \in J. \tag{3.2}$$

We say that the Mangasarian - Fromovitz Constraint Qualification (MFCQ) is fulfilled at the feasible point $\bar{x} \in M$ if there exists $\xi \in \mathbb{R}^n$ such that

$$Dg_j(\bar{x})\xi < 0, \text{ for all } j \in J_0(\bar{x}).$$

We recall that LICQ implies MFCQ. It is well known (cf. [5]) that if MFCQ is fulfilled at a local minimizer $\bar{x}$ of P, then, necessarily $\bar{x}$ is a stationary point for the problem P. Furthermore, the set of Lagrange multipliers of P corresponding to $\bar{x}$, namely

$$\mathcal{M}(\bar{x}) = \left\{ \mu = \left(\mu_j\right)_{j \in J} \in \mathbb{R}^{|J|} \mid \mu_j \geq 0, \ j \in J \text{ and } \mu \text{ satisfies } (3.1) - (3.2) \right\}$$

is a compact convex polyhedron if and only if MFCQ is satisfied at $\bar{x}$ ([2]).

Assume $J = \{1, \ldots, s\}$. Denote $G := (g_1 - b_1, g_2 - b_2, \ldots, g_s - b_s)$. For a given (f, G) and a subset $U \subset \mathbb{R}^n$, we consider the following semi-norm on $C^2(U, \mathbb{R}) \times C^2(U, \mathbb{R}^s)$:

$$\begin{aligned} &\text{norm } \left[\left(f, G\right), U \right] \\ &= \sup_{x \in U, \phi \in \{f, g_j - b_j, j \in J\}} \max \left\{ |\phi(x)| + \sum_j \left| \frac{\partial \phi}{\partial x_j}(x) \right| + \sum_{i,j} \left| \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) \right| \right\}. \end{aligned}$$

For $x \in \mathbb{R}^n$ and $\delta > 0, B(x, \delta)$ denote the Euclidean ball in $\mathbb{R}^n$, centered at x with radius δ.

Definition 3.2.1 *([3]) Let $x \in M$, $x \in$ int X be a stationary point for P. Then, x is called strongly stable if for some $\bar{\delta} > 0$ and each $\delta \in (0, \bar{\delta}]$ there exists $\alpha > 0$ such that, whenever $(\tilde{f}, \tilde{G})$ satisfies norm $[(f - \tilde{f}, G - \tilde{G}), B(x, \bar{\delta})] \leq \alpha$, the ball $B(x, \delta)$ containts a stationary point that is unique in $B(x, \bar{\delta})$.*

Note that strong stability in particular implies the continuous dependence of a stationary point on C^2 perturbations of the problem data. For $\bar{x} \in \mathbb{R}^n$ and $\bar{J} \subset J$ we put $T(\bar{x}, \bar{J}) = \bigcap_{j \in \bar{J}} \ker Dg_j(\bar{x})$. Moreover, given μ we define

$$L(x, \mu) = f(x) + \sum_{j \in J} \mu_j (g_j(x) - b_j).$$

Theorem 3.2.1 ([3]) *(Characterization of strong stability) Let $\bar{x} \in M$ be a stationary point for P.*

(i) Let LICQ be satisfied at $\bar{x}$ and let μ be the Lagrange parameter vector. Then, $\bar{x}$ is a strongly stable stationary point if and only if the Hessian $D_x^2 L(\bar{x}, \mu)$ has non-vanishing determinants with a common sign on the subspaces $T(\bar{x}, \tilde{J})$ with $J_+(\bar{x}) \subset \tilde{J} \subset J_0(\bar{x})$, where

$$J_+(\bar{x}) = \{ j \in J_0(\bar{x}) \mid \mu_j > 0 \}. \tag{3.3}$$

(ii) Let MFCQ be satisfied at $\bar{x}$, but not LICQ. Then, $\bar{x}$ is a strongly stable stationary point if and only if for every $\mu \in \mathcal{M}(\bar{x})$ the Hessian $D_x^2 L(\bar{x}, \mu)$ is positive definite on the subspace $T(\bar{x}, J_+(\bar{x}))$, with $J_+(\bar{x})$ as in (3.3).

3.3 Local convexification

The p-power formulation:

$$P_p : \min \left\{ [f(x)]^p \mid [g_j(x)]^p \le b_j^p, j \in J, x \in X \right\}, p > 0.$$

In the sequel we consider $\bar{x} \in \text{int } X$.

Remark 3.3.1 *$\bar{x}$ is a local minimizer of P if and only if $\bar{x}$ is a local minimizer of P_p.*

The Lagrangian function for P_p :

$$L_p(x, \gamma) = [f(x)]^p + \sum_{j \in J} \gamma_j \left([g_j(x)]^p - b_j^p \right).$$

Remark 3.3.2 $J_0(\bar{x}) = J_{p_0}(\bar{x})$, *where* $J_{p_0}(\bar{x}) = \left\{ j \in J \mid [g_j(\bar{x})]^p = b_j^p \right\}$.

Remark 3.3.3 *If MFCQ is satisfied for P at $\bar{x}$, then MFCQ is satisfied for P_p at $\bar{x}$. If LICQ is not fulfilled for P at $\bar{x}$ then LICQ is not fulfilled for P_p at $\bar{x}$.*

Remark 3.3.4 *If $\bar{x}$ is a local minimizer of P_p and MFCQ is fulfilled at $\bar{x}$, then there exist multipliers $\gamma_j \geq 0, j \in J$ such that:*

$$[f(\bar{x})]^{p-1} Df(\bar{x}) + \sum_{j \in J} \gamma_j [g_j(\bar{x})]^{p-1} Dg_j(\bar{x}) = 0, \tag{3.4}$$

$$\gamma_j([g_j(\bar{x})]^p - b_j^p) = 0, j \in J. \tag{3.5}$$

Remark 3.3.5 *If MFCQ is satisfied at a feasible point $\bar{x}$ of P_p, then the set of Lagrange multipliers of P_p corresponding to $\bar{x}$, namely*

$$\Gamma(\bar{x}) = \{\gamma \in \mathbb{R}^s \mid \gamma \geq 0 \text{ and } \gamma \text{ satisfies } (3.4) - (3.5)\}$$

is a compact convex polyhedron. Furthermore, by comparing (3.1) with (3.4) we see that there is a one-to-one correspondence between $\mathcal{M}(\bar{x})$ and $\Gamma(\bar{x})$ given by:

$$\gamma_j = \begin{cases} \frac{[f(\bar{x})]^{p-1}}{[g_j(\bar{x})]^{p-1}} \mu_j, & j \in J_+(\bar{x}), \\ 0, & \text{otherwise.} \end{cases} \tag{3.6}$$

Theorem 3.3.1 (Theorem 2.1 of [4]) *Let $\bar{x}$ be a local minimizer of P. Assume that $J_+(\bar{x}) \neq \emptyset$, LICQ is satisfied at $\bar{x}$ and $D_x^2 L(\bar{x}, \bar{\mu})$ is positive definite on $T(J_+(\bar{x}))$, where $\bar{\mu}$ is the unique Lagrange multiplier vector corresponding to $\bar{x}$. Then there exists $q > 0$ such that $D_x^2 L_p(\bar{x}, \bar{\gamma})$ is positive definite for all $p > q$, where $\bar{\gamma}$ is obtained from (3.6) with $\mu_j = \bar{\mu}_j$.*

Remark 3.3.6 *From the proof of Theorem 3.3.1 it is clear that the same result is obtained if we replace the assumption "LICQ is satisfied at $\bar{x}$" by "$Dg_j(\bar{x}) \neq 0$, for all $j \in J_+(\bar{x})$".*

Theorem 3.3.2 *If $\bar{x}$ is a strongly stable local minimizer of P, then there exists $q > 0$, such that the Hessian $D_x^2 L_p(\bar{x}, \gamma)$ is positive definite for all $p > q$, for all $\gamma \in \Gamma(\bar{x})$.*

Proof. If MFCQ is satisfied at $\bar{x}$ then $Dg_j(\bar{x}) \neq 0$, for all $j \in J_+(\bar{x})$ since $\exists \xi \in \mathbb{R}^n : Dg_j(\bar{x})\xi < 0$, for all $j \in J_0(\bar{x})$. Then, by Theorem 3.2.1, Theorem 3.3.1 and Remark 3.3.6 we have that for each $\mu \in \mathcal{M}(\bar{x})$, there exists a positive number corresponding to μ, say $q(\mu) > 0$, such that

$$D_x^2 L_p(\bar{x}, \gamma) \text{ is positive definite for all } p > q(\mu), \tag{3.7}$$

where γ is the multiplier associated to μ (see(3.6)).

Define

$$q := \max \{q(\mu) \mid \mu \text{ is a vertex of } \mathcal{M}(\bar{x})\}.$$

Now, by Karatheodory's theorem, each $\mu \in \mathcal{M}(\bar{x})$ is a convex combination of the vertexes of $\mathcal{M}(\bar{x})$. Let be $\mu \in \mathcal{M}(\bar{x})$. Then, there exist $\alpha_i > 0, \mu^i$ vertex of $\mathcal{M}(\bar{x})$, $i = 1, \ldots, m$, $1 \leq m \leq n+1$, such that $\sum_{i=1}^m \alpha_i = 1$ and $\mu = \sum_{i=1}^m \alpha_i \mu^i$. Let be $p > q$ and γ the multiplier corresponding to μ by (3.6). Since $D_x^2 L_p(\bar{x}, \gamma) = \sum_{i=1}^m \alpha_i D_x^2 L_p(\bar{x}, \gamma^i)$ where γ^i are the multipliers corresponding to μ^i by (3.6) and $D_x^2 L_p(\bar{x}, \gamma^i), i = 1, \ldots, m$ are definite positive if follows that $D_x^2 L_p(\bar{x}, \gamma)$ is positive definite.$\square$

3.4 The duality theorem

Dual function of P_p :

$$\phi_p(\gamma) = \min_{x \in V(\bar{x})} L_p(x, \gamma),$$

where $V(\bar{x})$ denotes an open neighborhood of $\bar{x}$.

Dual problem of P_p :

$$D_p : \max_{\gamma \geq 0} \phi_p(\gamma)$$

Theorem 3.4.1 *(See Theorem 6.2.5 of [1]) A pair $(\bar{x}, \bar{\gamma})$ where $\bar{x}$ is a local minimizer of P_p and $\bar{\gamma} \geq 0$, is a saddle point for the Lagrangian function $L_p(x, \gamma)$ if and only if*

a. $L_p(\bar{x},\bar{\gamma}) = \min_{x\in V(\bar{x})} L_p(x,\bar{\gamma})(= \phi_p(\bar{\gamma}))$,
b. $[g_j(\bar{x})]^p \leq b_j^p, j \in J$, and
c. $\sum_{j\in J} \bar{\gamma}_j([g_j(\bar{x})]^p - b_j^p) = 0$.

Moreover, $(\bar{x},\bar{\gamma})$ is a saddle point if and only if $\bar{x}$ and $\bar{\gamma}$ are, respectively, local minimizer and maximizer to the primal and dual problems P_p and D_p with no duality gap, that is, with $[f(\bar{x})]^p = \phi_p(\bar{\gamma})$.

Theorem 3.4.2 *Let $\bar{x}$ be a strongly stable local minimizer of P with optimal value r and Lagrange multiplier set $\mathcal{M}(\bar{x})$. Then there exists $q > 0$, such that for all $p > q$ each $\bar{\gamma} \in \Gamma(\bar{x})$ is a local maximizer of the dual problem D_p*

with $r^p = \phi_p(\bar{\gamma})$ and $L_p(\bar{x},\bar{\gamma}) = \phi_p(\bar{\gamma})$.

Proof. By Theorem 3.3.2, there exists $q > 0$, such that for all $p > q$ and for all $\bar{\gamma} \in \Gamma(\bar{x})$ it holds that

a. $L_p(\bar{x},\bar{\gamma}) = \min_{x\in V(\bar{x})} L_p(x,\bar{\gamma})(= \phi_p(\bar{\gamma}))$,
b. $[g_j(\bar{x})]^p \leq b_j^p, j \in J$,
c. $\sum_{j\in J} \bar{\gamma}_j([g_j(\bar{x})]^p - b_j^p) = 0$.

Then, by Theorem 3.4.1, for each $\bar{\gamma} \in \Gamma(\bar{x}), (\bar{x},\bar{\gamma})$ is a saddle point of the Lagrangian function $L_p(x,\gamma)$ and it holds that each $\bar{\gamma} \in \Gamma(\bar{x})$ is a local maximizer of D_p with no duality gap with the primal problem P_p. Since P and P_p are equivalent problems, we have $r^p = \phi_p(\bar{\gamma}) = L_p(\bar{x},\bar{\gamma})$.□

Example 3.4.1 *In this example the point $\bar{x} = (0.75, 1)$ is a strongly stable local (global) minimizer at which MFCQ is fulfilled but not LICQ.*

$$\begin{aligned}
\min f(x_1,x_2) &= x_1x_2 - 2x_1 - 2x_2 + 5\\
s.t.\ g_1(x_1,x_2) &= 4x_1 - 2x_2 \leq 1,\\
g_2(x_1,x_2) &= x_2 \leq 1,\\
g_3(x_1,x_2) &= x_1 + x_2 \leq 1.75,\\
X &= \{(x_1,x_2) \mid 0.6 \leq x_1 \leq 1, 0.1 \leq x_2 \leq 1.1\}.
\end{aligned}$$

$(f,\ g_j,\ j = 1,2,3 > 0$ for all $x \in X)$.

$$\mathcal{M}(\bar{x}) = \left\{ (\mu_1,\mu_{2,}\mu_3) \;\middle|\; \begin{array}{r} -1 + 4\mu_1 + \mu_3 = 0,\\ -1.25 - 2\mu_1 + \mu_2 + \mu_3 = 0,\\ \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0. \end{array} \right\}$$

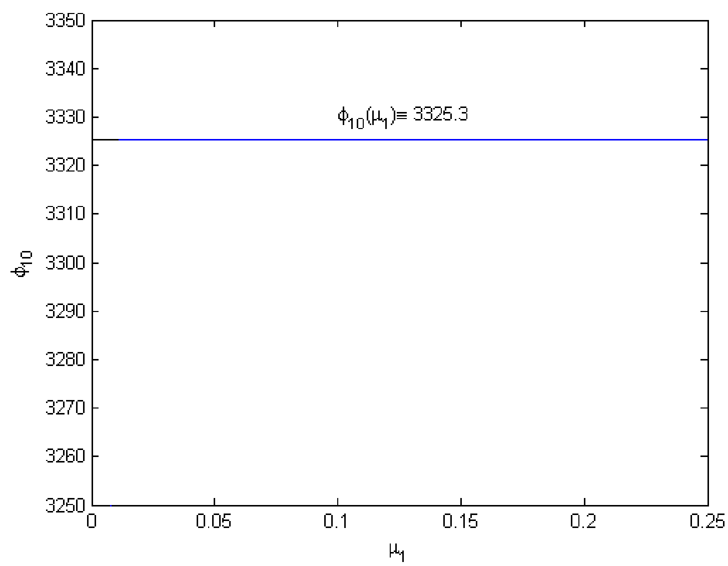

Figure 3.1: Bifurcation Diagram for $n = 3$ (radial case).

For example, take $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3) = (0.15, 1.15, 0.4) \in \mathcal{M}(\bar{x})$. Then we get $eig(D_x^2 L(\bar{x}, \bar{\mu})) = (-1, 1)$. That is, $D_x^2 L(\bar{x}, \bar{\mu})$ is indefinite.

For the problem P of this example the set $\mathcal{M}(\bar{x})$ has two vertexes, namely, $\bar{\mu}^1 = (0, 0.25, 1)$ and $\bar{\mu}^2 = (0.25, 1.75, 0)$. Following the notation of the proof of Theorem 3.3.2, $q(\bar{\mu}^1) = 9.8$, $q(\bar{\mu}^2) = 1.2$. Taking $q = \max\{q(\bar{\mu}^1), q(\bar{\mu}^2)\}$ we get $D_x^2 L_p(\bar{x}, \gamma)$ is positive definite for $p > q$, for all $\gamma \in \Gamma(\bar{x})$. Thus, for example, taking $p = 10$, for all $\bar{\gamma} \in \Gamma(\bar{x}), \phi_p(\bar{\gamma}) = 3325.3 = L_p(\bar{x}, \bar{\gamma}) = \min\{L_p(x, \bar{\gamma}) \mid x \in V(\bar{x})\}$ with $V(\bar{x})$ =int$([0.6, 1] \times [0.1, 1.1])$ and for each $\bar{\gamma} \in \Gamma(\bar{x})$, we can find a neighborhood $\mathcal{N}(\bar{\gamma})$ of $\bar{\gamma}$, such that $\phi_p(\bar{\gamma}) \geq \{\phi_p(\gamma) \mid \gamma \in \mathcal{N}(\bar{\gamma}), \gamma \geq 0\}$. See Figures 1 and 2 where for simplicity ϕ is calculated as a function of μ_1 and (μ_1, μ_2), respectively.

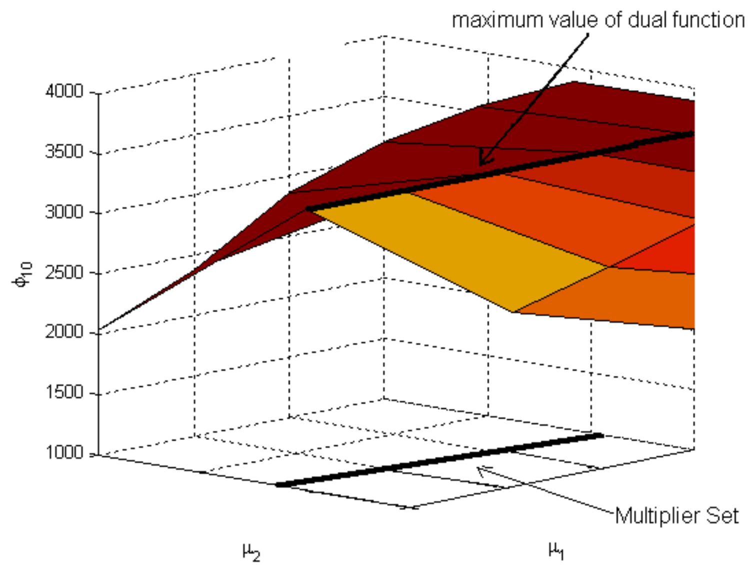

Figure 3.2: ϕ_{10} locally around $\mathcal{M}(\overline{x})$..

3.5 References

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4

On Local Saturation in Simultaneous Approximation

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Abstract: In this paper we survey some results of the authors on saturation in the setting of simultaneous approximation of functions by sequences of linear shape preserving operators. The theory of Tchebychev systems and generalized convexities play an important role. The results are applied to the derivatives of the classical Bernstein and Kantorovich operators.

4.1 Introduction

In the general setting of approximation of continuous functions by sequences of linear operators one usually deals with three topics, namely one studies the convergence of this type of processes, estimates the rate of convergence and checks the goodness of the estimates. As regards the first topics and under the hypothesis of positivity of the operators, the famous results of Bohman

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[3] and Korovkin [15] provided a powerful tool to study the convergence while the results of Newmann and Shapiro [23] and Shisha and Mond [24] showed the first general estimates of the degree of convergence. These estimates were stated in terms of the modulus of continuity of the function to be approximated, so they showed that the speed of convergence depends on the smoothness of the functions. In the opposite direction one finds the so called inverse results where, from the rate of convergence of certain approximation process one infers information about the regularity of the functions. A proper combination of both direct and inverse results guide us to the last aforesaid topic called saturation.

Many authors consider that the first saturation result was established by Voronovskaya [27] in 1932 for the classical Bernstein operators B_n:

Theorem 4.1.1 *If f is bounded on $[0,1]$, differentiable in some neighborhood of x, and has second derivative $f''(x)$ for some $x \in [0,1]$, then*

$$\lim_{n\to\infty} n(B_n f(x) - f(x)) = \frac{x(1-x)}{2} f''(x).$$

The theorem asserts that the convergence of $B_n f$ towards f cannot be too fast, even for very smooth functions; $B_n f(x) - f(x)$ is of order not better that $1/n$ if $f''(x) \neq 0$. Thus it is fixed a speed that in certain sense we may call optimal. Later on, some results appeared going deeper in this topic: DeLeeuw [11] proved that $|B_n f(x) - f(x)| = O(1/n)$ uniformly on each interval $[a,b] \subset (0,1)$ if and only if f is differentiable and its first derivative f' belongs to the class *Lip* 1, Bajsanski and Bojanić [2] proved in 1964 that if f is continuous on $[0,1]$ and $B_n f(x) - f(x) = o(1/n)$ for all $x \in (a,b) \subset (0,1)$, then f is linear in $[a,b]$, and that very year Lorentz [17] refined the result of DeLeeuw stating the following

Theorem 4.1.2 *Let $f \in C[0,1]$, then*

$$|B_n f(x) - f(x)| \leq \frac{Mx(1-x)}{2n} + o(1/n),\ 0 \leq x \leq 1,\ n = 1,2,\ldots$$

if and only if f' exists and belongs to $Lip_M 1$ on $[0,1]$

Roughly speaking, the notion of saturation appears when at the time of approximating functions by a sequence of linear operators, there exists a class of functions $\mathcal{S}$ for which the degree of approximation is 'optimal', and there is a second class, say $\mathcal{T}$, included in $\mathcal{S}$ whose functions present a better degree of approximation. Then the sequence is said to be saturated and $\mathcal{S}$ and $\mathcal{T}$ are called respectively the saturation class and the trivial class. Obviously the Bernstein operators are saturated, $\mathcal{T}$ is formed by the linear functions on $[0,1]$ and $\mathcal{S}$ is formed by these differentiable functions whose first derivative belongs to *Lip* 1 on $[0,1]$.

The proof of Theorem 4.1.2 based upon the construction of a proper functional associated to the sequence of operators. Though this technique was used in some other papers ([25], [26]), it could be applied only to particular cases and not to a general setting. On this respect, the aforesaid work of Bajsanski and Bojanić found further development in the papers of Amel'kovič [1] and Mühlbach [20] and finally in the outstanding one of Lorentz and Schumaker [19], a version of which we detail here (D^i will denote the usual i-th differential operator):

let w_0, w_1, w_2 be continuous, strictly positive functions on a real interval (a,b) such that $w_i \in C^{2-i}(a,b)$; from them we form the extended complete Tchebychev system $T = \{u_0, u_1, u_2\}$ and define the operators $\triangle_T$ and D_T as follows: let $c \in (a,b)$,

$$\begin{array}{l} u_0(t) = w_0(t) \\ u_1(t) = w_0(t) \int_c^t w_1(\alpha) d\alpha \\ u_2(t) = w_0(t) \int_c^t w_1(\alpha) \int_c^\alpha w_2(\beta) d\beta d\alpha \end{array}$$

$$\triangle_T g = \frac{1}{w_1} D^1 \left(\frac{1}{w_0} g \right), \quad D_T g = \frac{1}{w_2} D^1 \left(\frac{1}{w_1} D^1 \left(\frac{1}{w_0} g \right) \right),$$

Notice that $D_T u_2 = 1$ and $\{u_0, u_1\}$ is a fundamental system of solutions of the differential equation in the unknown z

$$D_T z = \frac{1}{w_2} D^1 \left(\frac{1}{w_1} D^1 \left(\frac{1}{w_0} z \right) \right) \equiv 0. \tag{4.1}$$

Theorem 4.1.3 *Assume that L_n is a sequence of linear operators that map $C[a,b]$ into itself. Assume in addition that the operators are positive and satisfy the following asymptotic formula whenever f belongs to the class C^2 in a neighborhood of a point $x \in (a,b)$:*

$$\lim_{n\to\infty} \lambda_n (L_n f(x) - f(x)) = p(x) D_T f(x), \tag{4.2}$$

where $\lambda_n > 0$ *converges to* $+\infty$ *with* n*,* $p(x) \geq 0$ *on* $[a, b]$ *and* $p(x)\, 0$ *on* (a, b)*.*

Then

$$\lambda_n |L_n f(x) - f(x)| \leq M p(x) + o(1), \;\; a < x < b,$$

if and only if $Mu_2 + f$ *and* $Mu_2 - f$ *are convex with respect to* u_0, u_1 *on* (a, b)

Notice that the definition of the operators $\triangle_T$ and D_T can be extended if one considers an integer $k > 2$ and $w_i \in C^{k-i}[a, b]$ for $i = 0, \ldots k$. Lorentz and Schumaker did it like this but they proved that a sequence of linear positive operators cannot satisfy a formula of the type (4.2) with $k > 2$. That is why we have restricted ourselves to the case $k = 2$.

Now our aim with this work is to survey some results obtained by the authors (see [9], [10], [13]) that extend the previous theorem in the sense that they are established under the framework of the so-called simultaneous approximation, that is to say, in the setting of the approximation of the derivatives of the functions by the corresponding derivatives of sequences of linear operators. As above we shall assume that the operators satisfy a shape preserving property (which generalizes positivity) and certain asymptotic formula. For the proofs, we maintain the same type of tools used in [19], namely the theory of Tchebychev systems as well as convexity arguments and certain Lipschitz conditions to characterize the saturation classes. We devote the following section to recall some of this elements we shall use later. Section 3 contains the main results while in the final one we present an application to the Bernstein operators.

It is important to point up that in the last ten years there has been an important progress in the setting of simultaneous approximation. We refer the reader to [22] , [21], [6], [7] to consult qualitative and quantitative results and to [8], [16] to read about asymptotic expressions. All these results represent the foundations on which the saturation results were established.

We end this introduction stating a general formal definition of saturation that extends the one stated by DeVore [12].

Definition 4.1.1 *Let* k *be a non negative integer, let* I *be a compact real interval, let* $(a, b) \subset I$ *and let* $L_n : C^k(I) \to C^k(I)$ *be a sequence of linear operators. The sequence* $D^k L_n$ *is said to be locally saturated on* (a, b)*, if* *there exists a sequence of function* $\phi_n = \phi_n(x)$ *which converges pointwise to* 0 *in* (a, b)*, a class of functions* $\mathcal{T}$ *such that* $f \in \mathcal{T}$ *if and only if*

$$\left|\frac{D^k L_n f(x) - D^k f(x)}{\phi_n(x)}\right| = o(1), \ x \in (a, b),$$

and a function $g \notin \mathcal{T}$*, such that*

$$\left|\frac{D^k L_n g(x) - D^k g(x)}{\phi_n(x)}\right| = \mathcal{O}(1). \tag{4.3}$$

ϕ_n *is called the saturarion order while* $\mathcal{T}$ *is called the trivial class. Besides the saturation class, denoted by* $\mathcal{S}$*, is that formed by those functions that satisfy (4.3).*

4.2 Generalized convexity and Lipschitz conditions

¿From the functions w_0, w_1, w_2 and u_0, u_1, u_2 considered in the previous section, we recover below some definitions and first results adapted to our purposes and related to different notions of convexity and Lipschitz conditions (for further details see [19], [4], [14]).

Proposition 4.2.1 *A continuous function* g *is convex on* (a, b) *with respect to* u_0, u_1 *if and only if for every* $a < t_1 < t_2 < b$ *the unique function* $\psi \in < u_0, u_1 >$ *interpolating* g *at* t_1, t_2 *satisfies* $\psi(t) \geq g(t)$ *for all* $t \in [t_1, t_2]$.

Definition 4.2.1 *A differentiable function* g *on* (a, b) *is said to satisfy a Lipschitz condition or belongs to the class* $Lip_M^T 1$ *with respect to the Tchebychev system* $T = \{u_0, u_1, u_2\}$ *with* $M \geq 0$ *whenever for* $a < t_1 < t_2 < b$

$$|\triangle_T g(t_2) - \triangle_T g(t_1)| \leq M \int_{t_1}^{t_2} w_2(s) ds. \tag{4.4}$$

Notice that if $w_2 = 1$, $g \in Lip_M^T 1$ amounts to the fact that $\triangle_T g$ belongs to the classical class $Lip_M 1$. If in addition $w_0 = w_1 = 1$, then it is equivalent to $D^1 g \in Lip_M 1$.

Next result connects the previous Lipschitz condition with convexity arguments and differential operators.

Lemma 4.2.1 *The following are equivalent for $g \in C[a,b]$ and $M \geq 0$:*

i) $Mu_2 + g$ *and* $Mu_2 - g$ *are convex with respect to* u_0, u_1 *in* (a,b),

ii) $\triangle_T g$ *is absolutely continuous in* (a,b) *and* $g \in Lip_M^T 1$ *in* (a,b),

iii) $|D_T g| \leq M$ *almost everywhere in* (a,b).

Now we recall a notion of convexity with respect to second order linear differential equations introduced by Bonsall [4].

Let $\mathcal{L}(y) = 0$ denote the differential equation

$$\mathcal{L}(y) \equiv D^2 y + a_1(t) D^1 y + a_0(t) y = 0.$$

Consider an interval (a,b) such that $\mathcal{L}(y) = 0$ has a unique solution, continuous in (a,b), taking any given real values y_1 and y_2 at any two given x_1 and x_2 within (a,b). Assume that $a_0(t), a_1(t)$ are continuous and differentiable whenever this is required.

Definition 4.2.2 *A real function f defined on (a,b) is said to be sub-$(\mathcal{L})$ in this interval if*

$$f(t) \leq S(f, x_1, x_2)(t)$$

for every t, x_1, x_2 such that $a < x_1 < t < x_2 < b$, $S(f, x_1, x_2)$ being the solution of $\mathcal{L}(y) = 0$ taking the values $f(x_1)$ and $f(x_2)$ at x_1 and x_2.

Proposition 4.2.2 *If f is sub-$(\mathcal{L})$ in (a,b), then it is continuous and there exist the right and left first derivatives $D^1_+ f$ and $D^1_- f$ in (a,b). Also $D^1_- f \leq D^1_+ f$ and indeed $D^1_- f = D^1_+ f$ except possibly in an enumerable set of points. Moreover, if for any $t_0 \in (a,b)$, $y = y(t)$ is a solution of $\mathcal{L}(y) = 0$ such that $y(t_0) = f(t_0)$, $D^1_- f(t_0) \leq y'(t_0) \leq D^1_+ f(t_0)$, then $y(t) \leq f(t)$ for all $t \in (a,b)$.*

Notice that, under the conditions of the previous section, a function is sub-(D_T) if and only it is convex with respect to u_0, u_1.

4.3 Saturation of k-convex operators

We consider a real compact interval I, a sequence L_n of linear operators mapping $C^k(I)$ into itself and fulfilling that:

(a) for each n, L_n is k-convex, i. e. $D^k f \geq 0$ implies $D^k L_n f \geq 0$ on I,

(b) there exist a sequence λ_n of real positive numbers which tends to infinity with n and two functions $p, q \in C^k(I)$, p being strictly positive on $Int(I)$, such that for each $f \in C^k(I)$, $k+2$-times differentiable in some neighborhood of $x \in Int(I)$,

$$\lim_{n\to\infty} \lambda_n \left(D^k L_n f(x) - D^k f(x) \right) = D^k \left(q D^1 f + p D^2 f \right)(x). \tag{4.5}$$

Our objective is to check if the sequence $D^k L_n$ is saturated, and if it is so, to find the trivial class and the saturation class. We begin stating a lemma which in the literature is usually known as a localization result. Here and in the sequel we write e_i with $i = 0, 1, \ldots$ for the polynomial $e_i(t) = t^i$ and $\mathbb{P}_k$ for the space spanned by $e_0, \ldots, e_k$.

Lemma 4.3.1 *Let $h \in C^k(I)$ and $x \in Int(I)$. Assume that $D^k h \geq 0$ on certain neighborhood N_x of a point x. Then*

$$D^k L_n h(x) \geq 0 + o(\lambda_n^{-1}).$$

Proof. Let us define a function $w \in C^k(I)$, k-convex on I such that on certain neighborhood of x satisfies $D^k h = D^k w$. To this end let $x_1, x_2 \in N_x$ with $x_1 < x < x_2$ and let

$$w(t) = \begin{cases} p_1(t) & \min(I) \leq t < x_1 \\ h(t) & x_1 \leq t < x_2 \\ p_2(t) & x_2 \leq t \leq \max(I) \end{cases}$$

where $p_1, p_2 \in \mathbb{P}_k$, and for $j = 1, 2$ and $0 \leq i \leq k$, $D^i p_j(x_j) = h(x_j)$. As $D^k h(x_j) \geq 0$, then w is k-convex on the whole I.

Now $D^k(w - h)$ vanishes in a neighborhood of x, hence from the asymptotic formula (4.5)

$$D^k L_n w(x) - D^k L_n h(x) = o(\lambda_n^{-1}).$$

We end applying hypothesis (a) to the k-convex function w:

$$0 \leq D^k L_n w(x) = D^k L_n h(x) + o(\lambda_n^{-1}).$$

□

Next result is a first step to search for the trivial class. It is an immediate consequence of hypothesis (b) so we state it without proof.

Lemma 4.3.2 *If $f \in C^k(I)$ satisfies $D^k\left(qD^1 f + pD^2 f\right) = 0$ in certain neighborhood of the point $x \in Int(I)$, then*

$$D^k L_n f(x) - D^k f(x) = o(\lambda_n^{-1}).$$

Now we are about to state the key result of the paper where the notion of convexity introduced by Bonsall will play an important role. To use it we need a second order differential operator. Let us see where it comes from.

First of all we consider the equation (see the right-hand side of (4.5))

$$D^k\left(qD^1 y + pD^2 y\right) \equiv 0. \tag{4.6}$$

and observe that $e_0, e_1, \ldots, e_{k-1}$ are solutions. Indeed, it suffices to apply (b) to these functions and use that from (a) we have that $D^k L_n e_i = 0$. Thus, by using the variable change $z = D^k y$, this differential equation can be reduced to this one of second order

$$pD^2 z + (q + kD^1 p)D^1 z + \left(\frac{k(k-1)D^2 p}{2} + kDq\right) z \equiv 0. \tag{4.7}$$

Finally we define the linear operator $\tilde{\mathcal{L}}$ to be

$$\tilde{\mathcal{L}}z = pD^2 z + (q + kD^1 p)D^1 z + \left(\frac{k(k-1)D^2 p}{2} + kDq\right) z$$

and assume the existence of three strictly positive functions $\tilde{\omega}_0, \tilde{\omega}_1, \tilde{\omega}_2$ with $\tilde{\omega}_i \in C^{2-i}(Int(I))$ such that for each z whenever it has sense

$$\tilde{\mathcal{L}}z = \frac{1}{\tilde{\omega}_2} D^1\left(\frac{1}{\tilde{\omega}_1} D^1\left(\frac{1}{\tilde{\omega}_0} z\right)\right). \tag{4.8}$$

Theorem 4.3.1 *The following are equivalent for $g \in C^k(I)$ and $(a, b) \subset I$.*

- $D^k g$ *is sub-*$(\tilde{\mathcal{L}})$ *in* (a,b),
- *for each* $x \in (a,b)$, $D^k L_n g(x) \geq D^k g(x) + o(\lambda_n^{-1})$.

Proof. Let $x \in (a,b)$. If $D^k g$ is sub-$(\tilde{\mathcal{L}})$ in (a,b), we consider z, solution of $\tilde{\mathcal{L}} z \equiv 0$, fulfilling

$$z(x) = D^k g(x), \quad D^1 z(x) = D^1_- \left(D^k g \right)(x).$$

Then using Proposition 4.2.2, we have that $z(t) \leq D^k g(t)$ for all $t \in (a,b)$. Now, from Lemma 4.3.1, if we take $y \in C^k(I)$, solution of (4.6) in (a,b) such that $D^k y(t) = z(t)$ for all $t \in (a,b)$, we obtain that

$$D^k L_n y(x) \leq D^k L_n g(x) + o(\lambda_n^{-1}),$$

or equivalently

$$D^k L_n y(x) - D^k y(x) \leq D^k L_n g(x) - D^k g(x) + o(\lambda_n^{-1}).$$

We apply Lemma 4.3.2 to the function y and obtain the desired inequality: $D^k L_n g(x) \geq D^k g(x) + o(\lambda_n^{-1})$ for $x \in (a,b)$.

To prove the converse, let us suppose that $D^k g$ is not sub-$(\tilde{\mathcal{L}})$ in (a,b), then there exist $a < t_1 < x < t_2 < b$ such that

$$S \left(D^k g, t_1, t_2 \right)(x) < D^k g(x).$$

Taking any function $\tilde{w} \in C^k(I)$, we can find $\epsilon > 0$ such that

$$S \left(\epsilon D^k \tilde{w} + D^k g, t_1, t_2 \right)(x) < \left(\epsilon D^k \tilde{w} + D^k g \right)(x).$$

Indeed, if $D^k \tilde{w}(x) - S \left(D^k \tilde{w}, t_1, t_2 \right)(x) \geq 0$ we can choose any $\epsilon > 0$, otherwise we can choose it so that it satisfies

$$0 < \epsilon < \frac{D^k g(x) - S \left(D^k g, t_1, t_2 \right)(x)}{S \left(D^k \tilde{w}, t_1, t_2 \right)(x) - D^k \tilde{w}(x)}.$$

Now, if we consider a solution z_0 of $\tilde{\mathcal{L}} z \equiv 0$ strictly positive in (a,b) (notice that its existence is guaranteed from the assumptions on $\tilde{\mathcal{L}}$), then the function

$$\frac{\epsilon D^k \tilde{w} + D^k g - S \left(\epsilon D^k \tilde{w} + D^k g, t_1, t_2 \right)}{z_0}$$

is continuous on $[t_1, t_2]$, it vanishes at both end points of this interval and takes a positive value at x. Hence it reaches its maximum value, say m, at a point $\tilde{x} \in (t_1, t_2)$. Consequently

$$\left(\epsilon D^k \tilde{w} + D^k g\right)(\tilde{x}) = \left(S\left(\epsilon D^k \tilde{w} + D^k g, t_1, t_2\right) + m z_0\right)(\tilde{x}),$$

and for all $t \in (t_1, t_2)$

$$\left(\epsilon D^k \tilde{w} + D^k g\right)(t) \leq \left(S\left(\epsilon D^k \tilde{w} + D^k g, t_1, t_2\right) + m z_0\right)(t).$$

Applying Lemma 4.3.1, taking $y_0, s \in C^k(I)$, solutions of (4.6) in (t_1, t_2) such that in this interval $D^k s = S\left(\epsilon D^k \tilde{w} + D^k g, t_1, t_2\right)$ and $D^k y_0 = z_0$, it is obtained that

$$\epsilon D^k L_n \tilde{w}(\tilde{x}) + D^k L_n g(\tilde{x}) - \left(\epsilon D^k \tilde{w}(\tilde{x}) + D^k g(\tilde{x})\right)$$

$$\leq D^k L_n s(\tilde{x}) + m D^k L_n y_0(\tilde{x}) - \left(D^k s(\tilde{x}) + m D^k y_0(\tilde{x})\right) + o(\lambda_n^{-1}),$$

from where if we apply Lemma 4.3.2 to the functions y_0 and s we derive that

$$D^k L_n g(\tilde{x}) - D^k g(\tilde{x}) \leq -\epsilon\left(D^k L_n \tilde{w}(\tilde{x}) - D^k \tilde{w}(\tilde{x})\right) + o(\lambda_n^{-1}).$$

Finally we get a contradiction if we recall the strict positivity of ϵ and take $\tilde{w} \in C^k(I)$ satisfying

$$D^k \tilde{w}(t) = \tilde{\omega}_0(t) \int_c^t \tilde{\omega}_1(\alpha) \int_c^\alpha \tilde{\omega}_2(\beta) d\beta d\alpha,$$

since from the asymptotic condition we have that $D^k L_n \tilde{w}(x) - D^k \tilde{w}(x) = 1 + o(\lambda_n^{-1})$.

$\square$

Now we write the main statement of the paper. Its proof is a very direct consequence of the previous result.

Theorem 4.3.2 *Let $M \geq 0$, let $a, b \in Int(I)$ with $a < b$ and let $f, W \in C^k(I)$. Then the following are equivalent:*

- $MD^kW + D^kf$ *and* $MD^kW + D^kf$ *are sub-*$(\tilde{\mathcal{L}})$ *in* (a,b),
- $|D^kL_nf(x) - D^kf(x)| \le M\left(D^kL_nW(x) - D^kW(x)\right) + o(\lambda_n^{-1})$, $x \in (a,b)$.

As a first corollary, now we are in a position to complete the information about the trivial class in the saturation problem of the sequence D^kL_n.

Corollary 4.3.1 *Let* $f \in C^k(I)$. *Then* $D^kL_nf(x) - D^kf(x) = o(\lambda_n^{-1})$ *for each* $x \in (a,b)$ *if and only if* f *is a solution of (4.6) in* (a,b). *Equivalently the trivial class in the saturation problem of the sequence* D^kL_n *is given by*

$$\mathcal{T} = \left\{ f \in C^k(I) : \; f_{|(a,b)} \textit{is solution of } D^k(qD^1y + pD^2y) \equiv 0 \right\}.$$

Proof. It suffices to apply Theorem 4.3.2 with $M = 0$ and observe that if D^kf and $-D^kf$ are sub-$(\tilde{\mathcal{L}})$ in (a,b), then D^kf is a solution of $\tilde{L}z \equiv 0$ in (a,b).

□

In addition to the previous result, different ways to choose the function W in Theorem 4.3.2 will lead us to different saturation results. We have chosen two of them in the following statements, the first one makes classical Lipschitz spaces appear, and the second one emphasizes the close relationship between asymptotic formulae and saturation results.

Corollary 4.3.2 *Let* $f \in C^k(I)$ *and* $(a,b) \subset I$. *Then*

$$\lambda_n \left| D^kL_nf(x) - D^kf(x) \right| \le M \cdot \tilde{\omega}_2^{-1}(x) + o(1), \; x \in (a,b),$$

if and only if

$$\tilde{\omega}_1 D^1(\tilde{\omega}_0^{-1} D^k f) \in Lip_M 1 \textit{ en } (a,b).$$

Proof.

Let $c \in (a,b)$ and let $W \in C^k(I)$, such that for all $t \in (a,b)$,

$$D^kW(t) = \tilde{\omega}_0(t) \int_c^t \tilde{\omega}_1(\alpha) \int_c^\alpha e_0(\beta) d\beta d\alpha, \; t \in (a,b), \tag{4.9}$$

Then it suffices to apply Theorem 4.3.2 with W as above, use hypothesis (b) taking into account that

$$D^k(qD^1W + pD^2W) = \tilde{\omega}_2^{-1}D(\tilde{\omega}_1^{-1}D(\tilde{\omega}_0^{-1}D^kW)) = \tilde{\omega}_2^{-1},$$

and finally use Lemma 4.2.1 with the extended complete Tchebychev system $T = \{u_0, u_1, u_2\}$ obtained from the strictly positive functions $\tilde{\omega}_0, \tilde{\omega}_1, e_0$, i.e.

$$u_0(t) = \tilde{\omega}_0, \quad u_1(t) = \tilde{\omega}_0(t)\int_c^t \tilde{\omega}_1(\alpha)d\alpha, \quad u_2(t) = D^kW(t).$$

$\square$

Corollary 4.3.3 *Let $f \in C^k(I)$ and $(a,b) \subset I$. Then*

$$\lambda_n \left| D^k L_n f(x) - D^k f(x) \right| \leq M + o(1), \ x \in (a,b),$$

if and only if

$$|D^k(qD^1f + pD^2f)| \leq M \quad \textit{almost everywhere in } (a,b).$$

Proof. Let $c \in (a,b)$ and let $W \in C^k(I)$ such that for all $t \in (a,b)$,

$$D^kW(t) = \tilde{\omega}_0(t)\int_c^t \tilde{\omega}_1(\alpha)\int_c^\alpha \tilde{\omega}_2(\beta)d\beta d\alpha, \ t \in (a,b).$$

It suffices to proceed as in the previous corollary observing now that

$$D^k(qD^1W + pD^2W) = \tilde{\omega}_2^{-1}D(\tilde{\omega}_1^{-1}D(\tilde{\omega}_0^{-1}D^kW)) = 1,$$

and considering the extended complete Tchebychev system $T = \{u_0, u_1, u_2\}$ obtained now from $\tilde{\omega}_0, \tilde{\omega}_1, \tilde{\omega}_2$.

$\square$

Finally the following two statements interpret the two previous results in terms of Definition 4.1.1.

Corollary 4.3.4 *The sequence D^kL_n is locally saturated in $(a,b) \subset I$ with order $\phi_n(x) = \frac{1}{\lambda_n \cdot \tilde{\omega}_2(x)}$ and saturation class given by*

$$\mathcal{S} = \left\{ f \in C^k(I) : \tilde{\omega}_1 D^1(\tilde{\omega}_0^{-1}D^kf) \in Lip_M 1 \ \textit{in } (a,b) \ \textit{for some } M \geq 0 \right\}.$$

Corollary 4.3.5 *The sequence $D^k L_n$ is locally saturated in $(a,b) \subset I$ with order $\phi_n(x) = \frac{1}{\lambda_n}$ and saturation class given by*

$$\mathcal{S} = \left\{ f \in C^k(I) : |D^k(qD^1 f + pD^2 f)| \leq M + o(1) \text{ a.e. in } (a,b) \text{ for some } M \geq 0 \right\}.$$

4.4 Applications

In this section we shall apply the previous results to the Bernstein operators referred in the introduction, defined by

$$B_n f(t) = \sum_{p=0}^{n} f\left(\frac{p}{n}\right) \binom{n}{p} t^p (1-t)^{n-p},$$

and to the Kantorovich operators, defined by

$$K_n f(t) = (n+1) \sum_{p=0}^{n} \binom{n}{p} t^p (1-t)^{n-p} \int_{\frac{p}{n+1}}^{\frac{p+1}{n+1}} f(z)dz.$$

As far as the conservative properties that these operators possess, it is only too well-known that B_n and K_n are k-convex for all $k \in \mathbb{N}_0$ (see e.g. [18]).

As far as (b) is concerned, both sequences fulfilled the asymptotic expression. The following table resumes the details contains references at this respect:

L_n	I	k	λ_n	$q(t)$	$p(t)$	References
B_n	$[0,1]$	$k = 0, 1, \ldots$	$2n$	0	$t(1-t)$	[27], [8], [16]
K_n	$[0,1]$	$k = 0, 1, \ldots$	$2(n+1)$	$1-2t$	$t(1-t)$	[5], [16]

Thus we are in conditions to apply the previous results to B_n and K_n. However it is necessary to comment firstly that it is an easy exercise to check that in both cases the existence of the functions $\tilde{\omega}_0, \tilde{\omega}_1, \tilde{\omega}_2$ which allow to represent the corresponding operator $\tilde{\mathcal{L}}$ as in (4.8) is guaranteed. They can be consulted in the following table (for $k = 1, 2, \dots$):

L_n	B_n	K_n
$\tilde{\omega}_0(t)$	$1/(t^{k-1})$	$1/t^k$
$\tilde{\omega}_1(t)$	$t^{k-2}/(1-t)^k$	$t^{2k}/(t(1-t))^{k+1}$
$\tilde{\omega}_2(t)$	$(1-t)^{k-1}$	$(1-t)^k$

Corollary 4.4.1 *Let $k \in \mathbb{N}$, $0 < a < b < 1$, $M \geq 0$, $p_B(t) = t(1-t)$ and $f \in C^k[0,1]$.*

- *$D^1 B_n f(x) - D^1 f(x) = o((2n)^{-1})$ for each $x \in (a,b)$ if and only if the restriction of f to the interval (a,b) belongs to the space*

$$\langle 1, t, \log(t^t(1-t)^{1-t})\rangle.$$

- *If $k > 1$, $D^k B_n f(x) - D^k f(x) = o(n^{-1})$ for each $x \in (a,b)$ if and only if the restriction of f to the interval (a,b) belongs to the space*

$$\langle 1, t, \dots t^{k-1}, \log t^t, \log(1-t)^{1-t}\rangle.$$

- $$2n\left|D^k B_n f(x) - D^k f(x)\right| \leq M\frac{1}{(1-x)^{k-1}} + o(1),\ x \in (a,b),$$

  if and only if

$$p_B^k\left(\frac{1}{e_{k-1}}D^{k+1}f + \frac{k-1}{e_k}D^k f\right) \in Lip_M 1 \ \textit{in}\ (a,b)$$

- $$2n\left|D^k B_n f(x) - D^k f(x)\right| \leq M + o(1),\ x \in (a,b),$$

  if and only if

$$|D^k(p_B D^2 f)| \leq M \quad \textit{a.e. in}\ (a,b).$$

Corollary 4.4.2 *Let $k \in \mathbb{N}$, $0 < a < b < 1$, $M \geq 0$, $q_K(t) = 1 - 2t$, $p_K(t) = t(1-t)$ and $f \in C^k[0,1]$.*

- $D^k K_n f(x) - D^k f(x) = o((2(n+1))^{-1})$ *for each* $x \in (a,b)$ *if and only if the restriction of* f *to the interval* (a,b) *belongs to the space*

$$\langle 1, t, t^{k-1}, (1+t)\log(1+t), t\log t\rangle.$$

-
$$2(n+1)\left|D^k K_n f(x) - D^k f(x)\right| \le M\frac{1}{(1-x)^k} + o(1),\ \ x \in (a,b),$$

if and only if

$$\frac{e_1}{(e_0-e_1)^{k+1}}D^{k+1}f + k(e_0-e_1)^{k+1}D^k f \in Lip_M 1 \ \ en\ (a,b).$$

-
$$2(n+1)\left|D^k K_n f(x) - D^k f(x)\right| \le M + o(1),\ \ x \in (a,b),$$

if and only if

$$|D^k\left(q_K D^1 f + p_K D^2 f\right)| \le M \ \ a.e.\ \ in\ (a,b).$$

4.5 References

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5

Butzer's problem and weighted modudi of smoothness

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Abstract We discuss several properties of sequences of positive linear operators which improve the rate of convergence of Bernstein sequences. In particular we characterize optimal sequences in terms of the norms of the operators in some Lipschitz-type spaces. Both, classical and weighted moduli of smoothness are considered. In the last section a new (non optimal) simple sequence which improves the rate of the Bernstein one is presented.

5.1 Introduction

Throughout the paper we denote by $C(I)$ the Banach (with the sup norm) space of all real continuous functions defined on $I=[0,1]$. For each positive integer n, Π_n denotes the family of all algebraic polynomials of degree not greater than n and, if $f\in C(I)$ and $n\in\mathbb{N}$, $E_n(f)=\inf\{\|f-p\|:p\in\Pi_n\}$.

Recall that, for $f\in C(I)$, $s\in(0,1]$ and $t\in(0,1/2]$ the (usual) first and second order modulus of smoothness are defined by

$$\omega_1(f,s)=\sup_{h\in(0,s]}\sup_{x,x+h\in I}|f(x+h)-f(x)|$$

and

$$\omega_2(f,t)=\sup_{h\in(0,t]}\sup_{x\pm h\in I}|f(x+h)-2f(x)+f(x-h)|,$$

respectively.

If $\alpha\in(0,2]$, we consider the space of Lipschitz functions defined as follows,

$$Lip^2_\alpha(I)=\{f\in C(I):\omega_2(f,t)=\mathcal{O}(t^\alpha)\}.$$

It is well known that if $f\in Lip^2_\alpha(I)$, then $E_n(f)=\mathcal{O}(n^{-\alpha})$. On the other hand, if $L_n:C(I)\to\Pi_n$ is a sequence of linear positive operators such that $\|L_n(e_i)-e_i\|\to 0$, $e_i(x)=x^i$ for $i\in\{0,1,2\}$, then there exists a constant C such that, for all $f\in C(I)$

$$\|L_n(f)-f\|\le C\omega_2\left(f,\frac{1}{\sqrt{n}}\right). \tag{5.1}$$

Notice that, for Lipschitz functions, the order of the best approximation $E_n(f)$ can not be derived from the last equation. This remark motivates a question (we will refer to this question as the general problem): does there exist a sequence $\{L_n\}$ of positive linear polynomial operator as above such that, for each $\alpha\in(0,2]$ and every $f\in Lip^2_\alpha(I)$, $\|L_n(f)-f\|=\mathcal{O}(n^{-\alpha})$. The particular case when each operator L_n is discretely defined is known as the Butzer problem (see [2]). The Butzer problem has been studied by several authors (see [3], [5], [6], [7], [8], [9], [10], [13] and [14]). In particular, the first solution was given in [3].

In order to motivate this paper let us present some remarks related with the problem above. Each solution of the general problem generates different solutions to the Butzer problem. Thus, if we do not ask for complementary facts, the discrete solution is always obtained as a consequence (see Theorem 5 in [10]).

As we will show below, for a sequence $\{L_n\}$ of positive linear operators one has that, for each $\alpha \in (0,2]$ and every $f \in Lip^2_\alpha(I)$, $\|L_n(f) - f\| = \mathcal{O}(n^{-\alpha})$ if and only if it assertion holds for the test function $e_i(x) = x^i$, for $i \in \{0,1,2\}$.

The general problem, as it is stated above, does not ask for a clear estimate of the rate of convergence. If a sequence $\{L_n\}$ solves the general problem, can we obtain an equation similar to (5.1), valid for functions $f \in Lip^2_\alpha(I)$, for which the estimate $\|f - L_n f\| = \mathcal{O}(n^{-\alpha})$ can be derived? Or can we obtain good estimation of the constants related with the big $\mathcal{O}$?

It is known that there exist functions $f \in C(I) \setminus Lip^2_\alpha(I)$ for which $E_n(f) = \mathcal{O}(n^{-\alpha})$. Therefore the following question arrives in a natural way: If a sequence $\{L_n\}$ solves the general problem, for which other functions $f \in C(I)$ does the estimate $\|f - L_n f\| = \mathcal{O}(n^{-\alpha})$ hold? The functions for which $E_n(f) = \mathcal{O}(n^{-\alpha})$ can be characterized by means of weighted moduli of smoothness. Now an associated question is the following: if a sequence $\{L_n\}$ solves the general problem and a function $f \in C(I)$ satisfies a Lipschitz type condition of order α with respect to a weighted modulus of smoothness, is it true that $\|f - L_n f\| = \mathcal{O}(n^{-\alpha})$. Of course, this last question depends on the specific weight we use.

It was stated in [12] a strong version of the Butzer problem. That is, does there exist a sequence $\{L_n\}$ of linear positive polynomial operator such that, for each $f \in C(I)$ and all $x \in I$,

$$| L_n(f,x) - f(x) | \leq C\omega_2\left(f, \frac{\sqrt{x(1-x)}}{n}\right)? \tag{5.2}$$

Characterization of the solutions of both problems are given found in [10].

In the second section of this paper we present other characterizations of the solution of the Butzer problem. In particular we show that: 1) the solution of the Butzer problem are related to the behavior of operators

$L_n - I : Lip^2_\alpha(I) \to C(I)$ (I the identity operator) where $Lip^2_\alpha(I)$ is provided with a Lipschitz-type norm, 2) if the functions $L_n(e_i)$, $i \in \{0, 1, 2\}$ are polynomials of degree not greater than two, then some strong inequalities hold. In the third section we analyze the Butzer problem for the case of the weighted modulus of smoothness. This allows to show that the estimate $\|L_n(f) - f\| = \mathcal{O}(n^{-\alpha})$ holds for some functions which are not in $Lip^2_\alpha(I)$. Finally, in the last section we present new sequences of positive linear operators with improves the rate of convergence given by the Bernstein one. The construction is very simple, but the new operators do not provide the optimal rate of convergence.

5.2 Characterization of the solutions

For a fixed sequence $\{L_n\}$ of operators, $L_n : C(I) \to C(I)$ and $x \in I$ we denote

$$A_n(x) =\mid L_n(e_0, x) - 1 \mid, \quad B_n(x) =\mid L_n(e_1, x) - x \mid$$

and

$$C_n(x) = L_n((e_1 - x)^2, x).$$

Our first result is an extension of Theorem 1 in [10]. There the theorem was established in terms of the Butzer problem and it was assumed that the operators can be written in the form

$$L_n(f, x) = \sum_{k=0}^{n} f(x_{n,k})\psi_{n,k}(x), \tag{5.3}$$

where $0 \leq x_{n,0} < x_{n,1} < \cdots < x_{n,n} \leq 1$ and $\psi_{n,k} : I \to \mathbb{R}^+$ is a family of basic functions. But, as it can be seen, this last condition was not used in the proof. Moreover, it is not important whether or not $L_n(f)$ is a polynomial, for $f \in C(I)$. Since our proof follows step by step the one given in [10], we omit it.

Theorem 5.2.1 *For a sequence $\{L_n\}$ of linear positive operators, $L_n : C(I) \to C(I)$, the following conditions are equivalent:*

(i) *For each $\alpha \in (0, 2]$ and every $f \in Lip^2_\alpha(I)$, $\|L_n(f) - f\| = \mathcal{O}(n^{-\alpha})$.*

(ii) *There exist constants C_i, $i \in \{0,1,2\}$, such that, for each $n \in \mathbb{N}$,*

$$\|L_n(e_i) - e_i\| \leq \frac{C_i}{n^2}, \quad i \in \{0,1,2\}. \tag{5.4}$$

We will show that, from condition (ii) in the last theorem, it can be obtained an explicit expression for the rate of the convergence. Some notations and previous results are needed.

For each $\alpha \in (0,2]$, the family $Lip_\alpha^2(I)$ becomes a Banach space with the norm

$$\|f\|_\alpha = \|f\| + \theta_\alpha(f),$$

where

$$\theta_\alpha(f) = \sup_{t\in(0,1/2]} \frac{\omega_2(f,t)}{t^\alpha}.$$

The norm of a linear bounded operator $M : Lip_\alpha^2(I) \to C(I)$ is denoted by $\|M\|_\alpha$.

In order to simplify, for $t \in (0,1/2]$, we denote

$$\lambda_\alpha(t) = \begin{cases} t & \text{if} \quad \alpha > 1 \\ t^\alpha & \text{if} \quad \alpha < 1. \\ t\ln(1/t) & \text{if} \quad \alpha = 1. \end{cases}$$

Theorem 5.2.2 *For each $\alpha \in (0,2]$ there exists a constant $C(\alpha)$ such that, for every $f \in Lip_\alpha^2(I)$ and $t \in (0,1/2]$,*

$$\omega_1(f,t) \leq C(\alpha)\|f\|_\alpha \lambda_\alpha(t).$$

Proof. The Marchaud inequality (see [15]) can be written in the form

$$\omega_1(f,t) \leq 8t\|f\| + \frac{t}{2}\int_t^{1/2} \frac{\omega_2(f,u)}{u^2}du,$$

whenever $f \in C[0,1]$ and $t \in (0,1/2]$.

In particular, if $f \in Lip_\alpha^2(I)$, for $\alpha \in (0,2]$, $\alpha \neq 1$, one has

$$\omega_1(f,t) \leq t\left(8\|f\| + \frac{\theta_\alpha(f)}{2}\int_t^{1/2} \frac{1}{s^{2-\alpha}}ds\right)$$

$$= t\left(8\|f\| + \frac{\theta_\alpha(f)}{2(1-\alpha)}\left[\frac{1}{t^{1-\alpha}} - 2^{1-\alpha}\right]\right) \leq \begin{cases} C\|f\|_\alpha t & \text{si} \quad \alpha > 1 \\ C\|f\|_\alpha t^\alpha & \text{si} \quad \alpha < 1. \end{cases}$$

If $\alpha = 1$, then

$$\omega_1(f,t) \leq t\left(8\|f\| + \frac{\theta_1(f)}{2}\int_t^{1/2}\frac{1}{s}ds\right)$$

$$= t\left(8\|f\| + \frac{\theta_1(f)}{2}\ln\frac{1}{2t}\right) \leq C\|f\|_1 t\ln(1/t).\blacksquare$$

Theorem 5.2.3 [11] *If $L : C[a,b] \to C[a,b]$ is a linear positive operator, then for $f \in C[a,b]$, $x \in [a,b]$ and $h \in (0,(b-a)/2]$ we have*

$$\mid f(x) - L(f,x) \mid \leq \mid 1 - L(e_0,x) \mid \|f\| + \tfrac{2}{h} \mid L((e_1 - x),x) \mid \omega_1(f,h)$$

$$+\tfrac{3}{4}\left[1 + L(e_0,x) + \mid L(e_0,x) - 1 \mid + \tfrac{2|L((e_1-x),x)|}{h} + \tfrac{L((e_1-x)^2,x)}{h^2}\right]\omega_2(f,h).$$

Theorem 5.2.4 *For a sequence $\{L_n\}$ of linear positive operators, $L_n : C(I) \to C(I)$, the following assertions are equivalent:*

(i) *For each $\alpha \in (0,2]$, there exist a constant $K(\alpha)$ such that, for $n \geq 2$*

$$\|L_n - I\|_\alpha \leq \frac{K(\alpha)}{n^\alpha}. \tag{5.5}$$

(ii) *There exist constants C_i, $i \in \{0,1,2\}$, such that, for each $n \in \mathbb{N}$,* (5.4) *holds.*

Proof. Assume that (5.4) holds. Then

$$B_n(x) = \mid L_n(e_1,x) - x + x(1 - L_n(e_0,x)) \mid \leq \frac{C_1 + C_2}{n^2}$$

and, from the identity $C_n(x) = L_n(e_2,x) - x^2 + x^2(L_n(e_0,x) - 1) + 2x(x - L_n(e_1,x))$, we have

$$C_n(x) \leq \frac{C_0 + 2C_1 + C_2}{n^2}.$$

It follows from Theorem 5.2.3, with $L = L_n(f,x)$ y $h = 1/n$ $(n \geq 2)$, and Theorem 5.2.2 that

$$\begin{aligned} \mid L_n(f,x) - f(x) \mid &\leq \|f\| A_n(x) + 2nB_n(x)\omega_1\left(f, \frac{1}{n}\right) \\ &+ \frac{3}{4}\left(2 + 2 \mid L(e_0,x) - 1 \mid + 2nB_n(x) + n^2 C_n(x)\right)\omega_2\left(f, \frac{1}{n}\right) \\ &\leq \frac{C_0\|f\|}{n^2} + \frac{2C_1}{n}\omega_1\left(f, \frac{1}{n}\right) + \frac{3}{2}\left(1 + \frac{C_0}{n^2} + \frac{C_1}{n} + C_2\right)\omega_2\left(f, \frac{1}{n}\right) \\ &\leq \frac{C_0\|f\|}{n^2} + \frac{2C_1 C(\alpha)\|f\|_\alpha}{n^\alpha} + \frac{3}{2}\left(1 + \frac{C_0}{n^2} + \frac{C_1}{n} + C_2\right)\frac{1}{n^\alpha}\theta_\alpha(f) \\ &\leq \left\{C_0 + 2C_1 C(\alpha) + \frac{3}{2}\left(1 + \frac{C_0}{n^2} + \frac{C_1}{n} + C_2\right)\right\}\|f\|_\alpha \frac{1}{n^\alpha}, \end{aligned}$$

where $C(\alpha)$ is taken as in Theorem 5.2.2. Thus (5.4) holds.

Finally, if we assume (5.5) (for all $\alpha \in (0,2]$), since $e_0, e_1, e_2 \in Lip_2(I)$ and

$$\|e_0\|_2 = 1, \quad \|e_1\|_2 = 1 \quad \|e_2\|_2 = 3,$$

we obtain (5.4).■

Remark 5.2.1 Under the conditions of Theorem 5.2.4, for $f \in Lip^2_\alpha(I)$,

$$\|L_n(f) - f\| \leq \frac{K(\alpha)\|f\|_\alpha}{n^\alpha}.$$

Moreover, from Theorems 5.2.2 and 5.2.3 we may obtain a good estimate of the constant $K(\alpha)$ in the last inequality, but we are no sure if it should be the better one.

Remark 5.2.2 In the references quoted in the Introduction there are examples of sequences, of linear positive polynomial operators $\{L_n\}$, for which (5.4) holds. Moreover, some of those operators preserve linear functions. That is $A_n(x) = B_n(x) = 0$. For these last operators the estimate in the first part of Theorem 5.2.4 can be rewritten as

$$\|L_n - I\|_\alpha \leq \frac{3(1 + C_2)}{2n^\alpha}.$$

In the last section of this paper we construct a sequence of linear positive polynomial operators such that $A_n(x) = B_n(x) = 0$ and $C_2 = 1$.

Our next result shows that, if we assume some restriction for the test functions $L_n(e_i)$, then conditions (5.4) imply a very stronger one. If particular we suppose each function $L_n(e_i)$ is a polynomial of degree not greater than two.

Theorem 5.2.5 *Let $\{L_n\}$, $L_n : C(I) \to C(I)$ be a sequence of positive linear operators such that, for $i \in \{0,1,2\}$, $L_n(e_i) \in \Pi_2$. Then, the following assertions are equivalent:*

(i) *For $i \in \{0,1,2\}$, $L_n(e_i, 0) = e_1(0)$, $L_n(e_i, 1) = e_i(1)$ and there exist constants C_i such that, for $n \in \mathbb{N}$, equations* (5.4) *holds.*

(ii) *For $i \in \{0,1,2\}$ there exist constants D_i such that, for $n \in \mathbb{N}$ and $x \in [0,1]$,*

$$| L_n(e_i, x) - x^i | \leq D_i \frac{x(1-x)}{n^2}.$$

Proof. If we write $L_n(e_i, x) = a_{n,i,0} + a_{n,i,1}x + a_{n,i,2}x^2$, then from the conditions $L_n(e_i, 0) = e_i(0)$ and $L_n(e_i, 1) = e_i(1)$, one has

$$L_n(e_0, x) = 1 - a_{n,0,2}x(1-x), \quad L_n(e_1, x) = x + a_{n,1,2}x(1-x)$$

and

$$L_n(e_2, x) = x^2 + a_{n,2,1}x(1-x).$$

Therefore

$$\|L_n(e_0) - 1\| = \frac{| a_{n,0,2} |}{4}, \quad \|L_n(e_1) - e_1 1\| = \frac{| a_{n,1,2} |}{4}$$

and

$$\|L_n(e_2) - e_2\| = \frac{| a_{n,2,1} |}{4}.$$

Thus (5.4) holds if and only if $a_{n,0,2} = \mathcal{O}(n^{-2})$, $a_{n,1,2} = \mathcal{O}(n^{-2})$ and $a_{n,2,1} = \mathcal{O}(n^{-2})$. But this is equivalent to (ii).■

Remark 5.2.3 There are sequences of operators which satisfies (5.4) and does not interpolate the test functions at the end point of the interval (see [4] and [10]).

5.3 Weighted modulus of smoothness

Let $\Omega(I)$ be the class of all continuous functions $\varphi : I \to I$, such that for $x \in (0,1)$, $\varphi(x) > 0$ and there exist $\gamma, \rho \geq 0$ for which $\varphi(x) \sim x^\gamma$ as $x \to 0$ and $\varphi(x) \sim (1-x)^\rho$ as $x \to 1$.

For $\varphi \in \Omega$ and functions $f \in C(I)$ the (Ditzian-Totik) weighted moduli of smoothness are defined by

$$\omega_1^\varphi(f,s) = \sup_{h\in(0,s]} \sup_{x\pm h\varphi(x)/2\in I} |f(x+h\varphi(x)/2) - f(x-h\varphi(x)/2)|$$

and

$$\omega_2^\varphi(f,t) = \sup_{h\in(0,t]} \sup_{x\pm h\varphi(x)\in I} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|\,.$$

Lipschitz-type functions and Lipschitz-type spaces are defined in a similar way to those associated with the usual modulus of smoothness. That is, if $\varphi \in \Omega(I)$ and $\alpha \in (0,2]$, then

$$Lip^2_{\varphi,\alpha}(I) = \{f \in C(I) : \omega_2^\varphi(f,t) = \mathcal{O}(t^\alpha)\},$$

and $\|f\|_{\varphi,\alpha} = \|f\| + \theta_{\varphi,\alpha}(f)$, where

$$\theta_{\varphi,\alpha}(f) = \sup_{t\in(0,1/2]} \frac{\omega_2^\varphi(f,t)}{t^\alpha}. \tag{5.6}$$

We need the following result from [1].

Theorem 5.3.1 *Fix $\varphi : [0,1] \to \mathbb{R}$ be a nonnegative continuous function which is strictly positive on $(0,1)$ and such that φ^2 is concave. Let $L : C[0,1] \to C[0,1]$ be a positive linear operator such that $L(e_0) = e_0$ and $L(e_1) = e_1$. For $f \in C[0,1]$ and $x \in (0,1)$, one has*

$$|f(x) - L(f,x)| \leq \left(\frac{3}{2} + \frac{3}{2(h\varphi(x))^2}(L(e_2,x) - x^2)\right)\omega_2^\varphi(f,h), \tag{5.7}$$

for $h \in (0, h_\varphi]$, where $h_\varphi = (2\varphi(1/2))^{-1}$.

For our next result notice that, if φ^2 is concave then, for $x \in (0,1)$,

$$\frac{\sqrt{x(1-x)}}{\varphi(x)} \leq \frac{\sqrt{(1/2)(1-(1/2))}}{\varphi(1/2))} = \frac{1}{2\varphi(1/2)}. \tag{5.8}$$

Theorem 5.3.2 *Fix $\varphi : [0,1] \to \mathbb{R}$ be a nonnegative continuous function which is strictly positive on $(0,1)$ and such that φ^2 is concave. Let $\{L_n\}$, $L_n : C(I) \to C(I)$ be a sequence of linear positive operators such that $L_n(e_i) \in \Pi_2$, $i \in \{0,1,2\}$. Then, the following assertions are equivalent.*

(i) *The sequence $\{L_n\}$ preserves the linear functions and there exists a constant C such that, for $x \in [0,1]$,*

$$L_n((e_1-x)^2, x) \leq \frac{Cx(1-x)}{n^2}. \tag{5.9}$$

(ii) *For $n \in \mathbb{N}$, $L_n(e_2,0) = 0$ and $L_n(e_2,1) = 1$ and there exist constants M and $c = c(\varphi)$ such that, for every $n \geq 1/c$ and $f \in C[0,1]$*

$$\|L_n(f) - f\| \leq M\omega_2^{\varphi}\left(f, \frac{1}{n}\right). \tag{5.10}$$

Proof. Assume that conditions in (i) hold. Since L_n preserves linear functions, it follows from (5.9) that $L_n(e_2,0) = 0$ and $L_n(e_2,1) = 1$. Let $c = c(\varphi)$ be as in Theorem 5.3.1 and fix $f \in C(I)$ and $x \in (0,1)$. It follows from Theorem 5.3.1, with $\Lambda = L_n$ and $h = 1/n$ and (5.8), that

$$\begin{aligned} \mid L_n(f,x) - f(x) \mid &\leq \frac{3}{2}\left\{1 + \frac{n^2 C_n(x)}{\varphi^2(x)}\right\}\omega_2^{\varphi}\left(f, \frac{1}{n}\right) \\ &\leq \frac{3}{2}\left\{1 + \frac{C}{4\varphi^2(1/2)}\right\}\omega_2^{\varphi}\left(f, \frac{1}{n}\right). \end{aligned} \tag{5.11}$$

Assume that (ii) holds. For $f \in \Pi_1$, $\omega_2^{\varphi}(f,t) = 0$. Thus $L_n(e_i) = e_i$, for $i \in \{0,1\}$. That is, L_n preserves linear functions. Finally, since $e_2 \in Lip^2_{\varphi,2}(I)$ and $\theta_{\varphi,2}(e_2) \leq 2$ (see (5.6)),

$$B_n(x) = \mid L_n(e_2,x) - x^2 \mid \leq M\omega_2^{\varphi}\left(e_2, \frac{1}{n}\right) \leq 2M\frac{1}{n^2}.$$

Then it follows from Theorem 5.2.5 that (5.9) holds.*square*

Remark 5.3.1 Examples of sequences $\{L_n\}$ satisfying conditions (i) in Theorem 5.3.2 where given in [7], [8] and [9].

Remark 5.3.2 It follows from Theorem 10 of [10] that, if $\{M_n\}$ is a sequence of positive linear operators which preserve linear functions and satisfy (5.9), then we can construct a sequence of positive linear operators $\{L_n\}$ such that (5.2) holds. That is we have (ii) in Theorem 5.3.2 for the particular case of the usual modulus of smoothness ($\varphi(x) \equiv 1$). Notice that our result gives better estimates. For instance, if $f(x) = x \ln x + (1-x)\ln(1-x)$, then $\omega_2(f,t) \sim t$, while $\omega_2^{\varphi}(f,t) \sim t^2$, for $\varphi(x) = \sqrt{x(1-x)}$. Then, from (5.2) we have $\mid L_n(f,x) - f(x) \mid \leq C\sqrt{x(1-x)}/n$ and Theorem 5.3.2 gives $\mid L_n(f,x) - f(x) \mid \leq C(x(1-x))/n^2$.

Remark 5.3.3 If we assume that the constant C in (5.9) is known, the last theorem gives also a good estimate of the constant related with equation (5.10).

For operators satisfying condition (i) in Theorem 5.3.2 we have also the analogous of Theorem 5.2.4 for weighted moduli of smoothness.

Theorem 5.3.3 *Let $\{L_n\}$, $L_n : C(I) \to C(I)$ be a sequence of linear positive operators which preserve linear functions and such that $L_n(e_2) \in \Pi_2$. Then, the following assertions are equivalent.*

(i) *For every nonnegative continuous function $\varphi : [0,1] \to \mathbb{R}$ which is strictly positive on $(0,1)$ and such that φ^2 is concave, there exist a constant $K(\varphi)$ such that, for each $\alpha \in (0,2]$ and every $n \in \mathbb{N}$, one has*

$$\|L_n - I\|_{\varphi,\alpha} \leq \frac{K(\varphi)}{n^{\alpha}}. \tag{5.12}$$

(ii) *For each $n \in \mathbb{N}$, L_n preserves the linear functions and there exists a constant C such that, for each $n \in \mathbb{N}$, (5.9) holds.*

Proof. Assume that (ii) holds and fix $f \in Lip^2_{\varphi,\alpha}(I)$. It follows from (5.7) (with $h = 1/n$, $n \geq 1/c(\varphi)$), (5.9) and (5.8) that, for $x \in (0,1)$,

$$| L_n(f,x) - f(x) | \leq \frac{3}{2}\left\{1 + \frac{C}{4\varphi^2(1/2)}\right\}\omega_2^{\varphi}\left(f, \frac{1}{n}\right) \leq$$

$$\leq \frac{3}{2}\left\{1 + \frac{C}{4\varphi^2(1/2)}\right\}\theta_{\varphi,\alpha}(f)\frac{1}{n^{\alpha}}.$$

On the other hand, if (5.12) holds, for $x \in [0,1]$,

$$| L_n(e_2,x) - x^2 | \leq K(\varphi)\theta_{\varphi,2}(e_2)\frac{1}{n^2} \leq 2K(\varphi)\frac{1}{n^2}.$$

and the result follows from Theorem 5.2.5.■

5.4 Examples

Let us present some examples where the ideas analyzed above can be applied.

Fix two real numbers α and β such that

$$2 - 3\alpha \geq 0, \tag{5.13}$$

$$0 < 8\alpha < \beta, \tag{5.14}$$

$$2(2-3\alpha)\beta + (4+\beta-8\alpha)(2\alpha-\beta) = 8\alpha + 4\alpha\beta - 16\alpha^2 - \beta^2 \geq 0. \tag{5.15}$$

For instance, these conditions hold if $\beta = 13\alpha$ and

$$0 < \alpha \leq \frac{8}{133}.$$

In fact, if $\beta = 13\alpha$, condition (5.15) can be rewritten as

$$26(2-3\alpha)\alpha - 11(4+5\alpha)\alpha = \alpha(8-133\alpha) \geq 0.$$

Notice that, from (5.13), (5.14) and (5.15) one has

$$3 + 2\beta - 4\alpha > 0. \tag{5.16}$$

Define

$$\lambda(x) = (2+\beta-4\alpha)x^2 - (4+\beta-8\alpha)x + 2 - 3\alpha$$

and

$$\gamma(x) = (2+\beta-4\alpha)x^2 - \beta x + \alpha.$$

One has $\lambda(0) = 2-3\alpha \geq 0$ and $\lambda(1) = \alpha \geq 0$. The function λ attains it minimum, on $[0,1]$, in the point

$$x_0 = \frac{4+\beta-8\alpha}{2(2+\beta-4\alpha)}.$$

But

$$\lambda(x_0) = \frac{(4+\beta-8\alpha)^2}{4(2+\beta-4\alpha)} - \frac{(4+\beta-8\alpha)^2}{2(2+\beta-4\alpha)} + 2 - 3\alpha$$

$$= -\frac{(4+\beta-8\alpha)^2}{4(2+\beta-4\alpha)} + 2 - 3\alpha = \frac{2(2-3\alpha)\beta + (4+\beta-8\alpha)(2\alpha-\beta)}{4(2+\beta-4\alpha)} \geq 0.$$

Thus, for $x \in [0,1]$,

$$\lambda(x) \geq 0. \tag{5.17}$$

On the other hand, $\gamma(0) = \alpha \geq 0$, $\gamma(1) = 2-3\alpha \geq 0$ and $\gamma'(x_1) = 0$ if and only if

$$x_1 = \frac{\beta}{2(2+\beta-4\alpha)}.$$

But

$$\gamma(x_1) = \frac{\beta^2}{4(2+\beta-4\alpha)} - \frac{\beta^2}{2(2+\beta-4\alpha)} + \alpha = -\frac{\beta^2}{4(2+\beta-4\alpha)} + \alpha$$

$$= \frac{8\alpha + 4\alpha\beta - 16\alpha^2 - \beta^2}{4(2+\beta-4\alpha)} \geq 0.$$

Therefore, for $x \in [0,1]$,

$$\gamma(x) \geq 0. \tag{5.18}$$

It follows from (5.17), (5.18) and (5.16) that the linear operator $L_4 : C[0,1] \to \Pi_4$,

$$L_4(f,x) = ((1-x)^4 + 4\alpha x(1-x)^3)f(0)$$

$$+2x(1-x)((2+\beta-4\alpha)x^2 - (4+\beta-8\alpha)x + 2 - 3\alpha)f\left(\frac{1}{4}\right)$$

$$+2(3+2\beta-4\alpha)x^2(1-x)^2 f\left(\frac{1}{2}\right) +$$

$$+2x(1-x)((2+\beta-4\alpha)x^2 - \beta x + \alpha)f\left(\frac{3}{4}\right)$$

$$+(x^4 + 4\alpha x^3(1-x))f(1)$$

is positive. One can prove that L_4 preserves linear functions and

$$L_4(e_2,x) - x^2 = \frac{x(1-x)}{4}\left(1 + 3\alpha - (\beta+4\alpha)x(1-x)\right). \tag{5.19}$$

Thus

$$\|L_4(e_2) - e_2\| = \frac{1}{16}\left(1 + \frac{8\alpha-\beta}{4}\right).$$

If $L_m : C[0,1] \to \Pi_m$ is a linear operator which reproduces linear functions and

$$L(f,x) = \sum_{i=0}^{m} Q_i(x) f(t_i), \tag{5.20}$$

where $t_i = i/m$ and $Q_i(x) \geq 0$ for $x \in [0,1]$ and $0 \leq i \leq m$, then we construct a new positive linear operator $L_{2m} : C[0,1] \to \Pi_{2m}$ by setting

$$L_{2m}(f,x) = \sum_{i=0}^{m} Q_i^2(x) f(t_i) + 2\sum_{i=0}^{m-1}\left(\sum_{j=i+1}^{m} Q_i(x)Q_j(x) f\left(\frac{t_i+t_j}{2}\right)\right).$$

Let us prove that L_{2m} reproduces linear functions and for $x \in [0,1]$,

$$0 \leq L^*(e_2,x) - x^2 = \frac{1}{2}\left(L(e_2,x) - x^2\right).$$

We start with the identities

$$1 = \left(\sum_{k=0}^{m} Q_k(x)\right)\left(\sum_{k=0}^{m} Q_k(x)\right) = \sum_{k=0}^{m} Q_k^2(x) + 2\sum_{i=0}^{m-1}\sum_{j=i+1}^{m} Q_i(x)Q_j(x),$$

$$x = \left(\sum_{k=0}^{m} Q_k(x)t_k\right)\left(\sum_{k=0}^{m} Q_k(x)\right) = \sum_{k=0}^{m} Q_k^2(x)t_k + \sum_{i=0}^{m-1}\sum_{j=i+1}^{m} Q_i(x)Q_j(x)(t_i+t_j),$$

$$x^2 = \left(\sum_{k=0}^{m} Q_k(x)t_k\right)\left(\sum_{k=0}^{m} Q_k(x)t_k\right) = \sum_{k=0}^{m} Q_k^2(x)t_k^2 + 2\sum_{i=0}^{m-1}\sum_{j=i+1}^{m} Q_i(x)Q_j(x)t_it_j.$$

and

$$\left(\sum_{k=0}^{m} Q_k(x)t_k^2\right)\left(\sum_{k=0}^{m} Q_k(x)\right) = \sum_{k=0}^{m} Q_k^2(x)t_k^2 + \sum_{i=0}^{m-1}\sum_{j=i+1}^{m} Q_i(x)Q_j(x)(t_i^2+t_j^2).$$

The first two equations given above show that L_{2m} preserves linear functions. One the other hand

$$L_{2m}(e_2,x) = \sum_{i=0}^{m} Q_i^2(x)t_i^2 + \sum_{i=0}^{m-1}\left(\sum_{j=i+1}^{m} Q_i(x)Q_j(x)\left(\frac{t_i}{2} + t_jt_k + \frac{t_j}{2}\right)\right)$$

$$= \frac{1}{2}\sum_{i=0}^{m} Q_i^2(x)t_i^2 + \frac{1}{2}L(e_2,x) + \sum_{i=0}^{m-1}\left(\sum_{j=i+1}^{m} Q_i(x)Q_j(x)t_jt_k\right) =$$

$$= \frac{1}{2}L_m(e_2,x) + \frac{1}{2}x^2.$$

Thus from the operator L_4 defined in (5.19) we obtain a sequence of positive linear operators $L_{4.2^n} : C[0,1] \to \Pi_{4.2^n}$ which reproduce linear functions and

$$\|L_{4.2^n}(e_2) - e_2\| = \frac{1}{2^n}\|L_4(e_2) - e_2\| = \frac{1}{16.2^n}\left(1 + \frac{8\alpha - \beta}{4}\right).$$

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6

Generalizations of Fredholm elements in Banach algebras

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Abstract: An Atkinson's theorem states that Fredholm operators are those operators T for which $\pi(T)$ is invertible, where π is the homomorphism $\pi : B(X) \to B(X)/F_0(X)$. In this note we present generalizations of Fredholm theory in Banach algebras replacing the usual invertibility by other notions of invertibility.

6.1 Introduction

In the early 20th century, I. Fredholm investigated and solved certain types of integral equations, which now we may call operational equations involving integral operators with continuous kernel. This investigations became the beginnings of what we now call Fredholm theory for bounded linear operators. Since the set of bounded linear operators forms a Banach algebra, it is natural to ask wheter there is an analogous theory for general Banach algebras. Several people have studied this question, from the papers of B.A Barnes ([2], [3]) to the monograph by B.A. Barnes, G.J. Murphy, M.R.F. Smyth and T.T. West ([5]), where the interested reader may find a detailed account, and the work of R. Harte ([15]). In this paper we will review some generalizations of the theory of Fredholm operators and their analogue in Banach algebras.

Let X be a Banach space. By $B(X)$ we will denote the set of bounded linear operators $T : X \to X$. For $T \in B(X)$, the range of T will be denoted by $R(T)$ and its kernel by $N(T)$.

Special attention will deserve the set of all compact operators and the set of all finite rank operators, denoted by $K(X)$ and $F_0(X)$ respectively.

For $T \in B(X)$, let us denote by $n(T)$ the dimension of $N(T)$, which is called the nullity of T, and by $d(T)$ the dimension of the quotient space $X/R(T)$, which is called the defect of T. These quantities enable us to define Fredholm operators. For a detailed account of the theory we are about to sketch, the interested reader might refer to [5] and [11].

We say that an operator $T \in B(X)$ is a *Fredholm operator* if:
(1) $n(T) < \infty$.
(2) $d(T) < \infty$.

It is a basic fact that $d(T) < \infty$ implies $R(T)$ is closed.

For a Fredholm operator it makes sense to consider the quantity $n(T) - d(T)$. This is known as the *index of a Fredholm operator* T, and will be denoted by $\mathrm{ind}(T)$.

Now we proceed towards the Atkinson's characterization of Fredholm operators, which will give us the main idea for further generalizations of Fredholm theory in Banach algebras.

Let $\pi_K : B(X) \to B(X)/K(X)$ be the natural homomorphism given by $\pi_K(T) = T + K(X)$, and let π_0 be the natural homomorphism of $B(X)$ into $B(X)/F_0(X)$ given by $\pi_0(T) = T + F_0(X)$.

Theorem 6.1.1 *(Atkinson's Characterization) Let $T \in B(X)$. The following statements are equivalent:*
(1) T is a Fredholm operator.
(2) $\pi_K(T)$ is invertible in the algebra $B(X)/K(X)$.
(3) $\pi_0(T)$ is invertible in the algebra $B(X)/F_0(X)$.

In order to understand the role of $F_0(X)$ we now turn to Banach algebras theory.

Let A be a Banach algebra with identity e. A subset J of A is a *left (right) ideal* if $AJ \subseteq J$ ($JA \subseteq J$). If J is a left and right ideal of A then J

is a two-sided ideal (or just an *ideal*). A left ideal J of A is called *regular* (or also modular) if there exists an element $b \in A$ such that $A(e-b) \subseteq J$. A two-sided ideal J of A is said to be *primitive* if there exists a maximal regular left ideal L of A such that $J = \{x \in A : xA \subseteq L\}$. The *radical* of A is defined as $\mathrm{rad}(A) = \bigcap\{P : P \text{ is a primitive ideal of } A\}$. If $\mathrm{rad}(A) = \{0\}$, then we say that A is *semisimple*.

Recall that J is a *minimal left ideal* of A if $J \neq \{0\}$ and it is such that $\{0\}$ and J are the only left ideals contained in J. If A is semisimple then the sum of minimal right ideals of A coincides with the sum of minimal left ideals of A, and in this case this sum will be called the *socle* of A, denoted $\mathrm{soc}(A)$. If A has no minimal ideals then $\mathrm{soc}(A) = \{0\}$. Since the socle is a sum of ideals, it is an ideal.

A *minimal idempotent* element of A is an idempotent p such that $p \neq 0$ and pAp is a division algebra. $\mathrm{Min}(A)$ will stand for the set of all minimal idempotents of A. If A is semisimple, then J is a minimal left ideal if and only if there exists a minimal idempotent p such that $J = Ap$ (see [8]). Also, for a semisimple Banach algebra A (see [1, §5.2]):

$$\mathrm{soc}(A) = \left\{ \sum_{k=1}^{n} Ap_k : n \in \mathbb{N}\ , p_k \in \mathrm{Min}(A) \right\}.$$

Returning to the bounded linear operators algebra, it is known that $\mathrm{rad}(B(X)) = \{0\}$, so $B(X)$ is semisimple. Also the minimal idempotents of $B(X)$ are the rank-one projections and thus, the socle is the ideal of all finite-dimensional operators, $F_0(X)$ ([1, Example 5.9]).

With Atkinson's characterization as motivation, the Fredholm elements of an algebra are defined in the following way:

Let A be a semisimple Banach algebra and $\pi : A \to A/\mathrm{soc}(A)$ the natural homomorphism. An element $a \in A$ is a *Fredholm element* if $\pi(a)$ is invertible in $(A/\mathrm{soc}(A))$. The set of Fredholm elements of A will be denoted by $\Phi(A)$.

If A is not semisimple, we can extend our theory by considering the quotient algebra $A' = A/\mathrm{rad}(A)$, which is semisimple. Since the socle may not exist, in its place we use the *presocle*: $\mathrm{psoc}(A) = \{x \in A : x' \in \mathrm{soc}(A')\}$, where x' stands for the coset $x + \mathrm{rad}(A)$. If A is semisimple, it is clear that $\mathrm{psoc}(A) = \mathrm{soc}(A)$.

In an arbitrary Banach algebra A with unit e, an element x is called a *Fredholm element* of A if there exists $y \in A$ such that $xy-e,\, yx-e \in \mathrm{psoc}(A)$ (see [5, Section F.3]).

In an unitary Banach algebra A, the *ideal of inessential elements* of A is defined to be:

$$I(A) = \bigcap \{P : P \text{ is a primitive ideal of } A \text{ with } \mathrm{psoc}(A) \subseteq P\}\,.$$

An ideal J of A such that $J \subseteq I(A)$ is called an *inessential ideal* of A. It is possible to develop a Fredholm theory relative to each inessential ideal J by defining $x \in A$ a *Fredholm element relative to* J if there exists $y \in A$ such that $xy - e,\, yx - e \in J$.

In particular, for $A = B(X)$ we have

$$\mathrm{soc}(B(X)) = F_0(X) \subseteq K(X) \subseteq I(B(X)),$$

and from Atkinson's characterization and [1, Corollary 5.6] the Fredholm operators are those which are invertible in $B(X)$ modulo every ideal of operators contained in $I(B(X))$. Recall $F_0(X)$ is contained in every ideal of $B(X)$. If A is semisimple and J is an ideal such that $\mathrm{soc}(A) \subseteq J \subseteq I(A)$, then J may be called a Φ-ideal.

In [4] B.A. Barnes developed a Fredholm and perturbation theory for elements of an algebra with respect to a Banach subalgebra.

6.2 Generalized Fredholm elements

For an algebra A with unit e, we will denote by A^{-1} the group of invertible elements in A. In order to introduce our first generalization of Fredholm elements we need some "generalized" notions of invertibility.

If A is an algebra, we say that $a \in A$ is *relatively regular* if there exists some $b \in A$ such that $a = aba$, and then such b is called a *generalized inverse* of a. It is easily checked that if b is a generalized inverse for a, then $c = bab \in A$ is such that $aca = a$ and $cac = c$. The set of all relatively regular elements of A will be denoted by $\overset{\sqcap}{A}$. In a semisimple Banach algebra, the elements of the socle are relatively regular ($\mathrm{soc}(A) \subseteq \overset{\sqcap}{A}$).

An element $a \in A$ is called *group invertible* if there exists $b \in A$ such that $a = aba$, $b = bab$ and $ab = ba$. In this case, b is the group inverse of a

and b is the group inverse of a. The group inverse is unique if it exists. The name "group" comes from the fact that $\{a, b\}$ generates an Abelian group (with identity ab).

It is well known that in the Banach algebra $B(X)$ the relatively regular elements are precisely those operators with closed range and whose range and kernel are complemented in the sense that there exist subspaces $A, B \subset X$ such that $X = R(T) \oplus A = B \oplus N(T)$. If T is a Fredholm operator, since $R(T)$ is finite-codimensional then $R(T)$ is complemented and since $N(T)$ is finite dimensional then $N(T)$ is complemented, and it follows that T has a generalized inverse. This observation leads to the definition of a generalized Fredholm element found in [5] as the relatively regular operators. However, the generalized inverse is not unique, so we use the approach of S.R. Caradus ([10]), which gives us nicer properties by defining the generalized Fredholm elements as follows:

Definition 6.2.1 (10) *$T \in B(X)$ is a generalized Fredholm operator if there is some $S \in B(X)$ such that $TST = T$ and $I - TS - ST$ is a Fredholm operator.*

In a series of papers ([19] to [20]), Ch. Schmoeger has investigated this class of operators. For $A, B \in B(X)$, if AB and BA are Fredholm operators, then A and B are Fredholm. Now, if AB is a Fredholm operator, BA need not to be Fredholm, however, BA is a generalized Fredholm operator (see [21]).

D. Männle and Ch. Schmoeger ([14]) introduced the generalized Fredholm elements for semisimple algebras:

Definition 6.2.2 (14) *If A is a semisimple algebra, then $a \in A$ is a generalized Fredholm element if there is some $b \in A$ such that $aba = a$ and $e - ab - ba$ is a Fredholm element of A.*

The set of generalized Fredholm elements in A will be denoted by $\Phi_g(A)$. Now some examples of generalized Fredholm elements (see [14]):

Example 6.2.1 *(1) The elements of the socle are generalized Fredholm elements.*

If $a \in \mathrm{soc}(A)$, then there is $b \in \mathrm{soc}(A)$ such that $aba = a$. Since $ab \in \mathrm{soc}(A)$ we have

$$\pi(e - ab - ba) = \pi(e) \in (A/\mathrm{soc}(A))^{-1},$$

thus

$$e - ab - ba \in \Phi(A).$$

(2) Every invertible element is a generalized Fredholm element.
Let $a \in A^{-1}$ and $b = a^{-1}$, then $aba = a$ and

$$e - ab - ba = -e \in A^{-1} \subseteq \Phi(A).$$

(3) The Fredholm elements are generalized Fredholm elements.
Let $a \in \Phi(A)$. Then $\pi(ab) = \pi(e) = \pi(ba)$ for each generalized inverse b of a. Hence

$$\pi(e - ab - ba) = -\pi(e) \in (A/\mathrm{soc}(A))^{-1},$$

and

$$e - ab - ba \in \Phi(X).$$

In the algebra $B(X)$ the set of compact operators $K(X)$ is an inessential ideal. It is easily checked that the zero operator O is a generalized Fredholm operator. Take an operator $K \in K(X)$ such that $R(K)$ is not finite-dimensional. Since $R(K)$ is not closed then the range of $O+K = K$ is not closed and thus it is not relatively regular. Therefore $O+K$ is not a generalized Fredholm operator, showing that $\Phi_g(B(X))+K(X) \nsubseteq \Phi_g(B(X))$.

In view of the above, is natural to expect that Φ-ideals different from $\mathrm{soc}(A)$ are not suitable to answer questions about generalized Fredholm operators. In fact, if J is an inessential ideal of an infinite-dimensional semisimple algebra A and if $\Phi_g(A) + J \subseteq \Phi_g(A)$ then $J = \mathrm{soc}(A)$ (see [14, Corollary 6.8]).

We have an Atkinson-type characterization for the generalized Fredholm elements (see [19]):

Theorem 6.2.1 *An element $a \in A$ is generalized Fredholm if and only if $\pi(a)$ is group invertible in $A/\mathrm{soc}(A)$.*

Proof. [14, Theorem 3.12] and [16, Theorem 2.3].□

From this, using the properties of group invertible elements, we get some algebraic properties of the generalized Fredholm elements:

Proposition 6.2.1 *Let $a, b \in \Phi_g(A)$.*
(1) If $ab - ba \in \mathrm{soc}(A)$, then $ab \in \Phi_g(A)$.
(2) If $n \in \mathbb{N}$, then $a^n \in \Phi_g(A)$.
(3) If $ab, ba \in \mathrm{soc}(A)$, then $a + b \in \Phi_g(A)$.
(4) If $x, y \in A$, then $e - xy \in \Phi_g(A)$ if and only if $e - yx \in \Phi_g(A)$.

Proof. (1),(2) [14, Theorem 5.2], (3) [14, Theorem 5.6], (4) [14, Theorem 5.3].□

If x is a Fredholm element of a Banach algebra then there exists $\delta > 0$ such that $\lambda e - x$ is a Fredholm element for $0 < |\lambda| < \delta$ (see [5, Theorem F.2.10]). For generalized Fredholm elements we have:

Theorem 6.2.2 (14, Theorem 3.11 (4)) *If A is a Banach algebra and $x \in \Phi_g(A)$, then there is $\delta > 0$ such that*

$$\lambda e - x \in \Phi_g(A) \text{ for all } \lambda \in \mathbb{C} \text{ satisfying } |\lambda| < \delta,$$

and

$$\lambda e - x \in \Phi(A) \text{ for all } \lambda \in \mathbb{C} \text{ satisfying } 0 < |\lambda| < \delta.$$

Using the above result, we can see that $\Phi_g(A) \subseteq \overline{\Phi(A)}$ ([14, Theorem 3.6]).

6.3 Drazin-Fredholm elements

M. Berkani introduced the B-Fredholm operators in [6], where they are defined as follows:

Definition 6.3.1 (6) *Let $T \in B(X)$ be an operator and $T_n : R(T^n) \to R(T^n)$ be the operator given by $T_n x = Tx$ for all $x \in R(T^n)$. We say that T is a B-Fredholm operator if there is some $n \in \mathbb{N}$ such that the range of T^n is closed and T_n is a Fredholm operator.*

The B-Fredholm operators have some nice properties. If T_n is a Fredholm operator then T_m is a Fredholm operator and $\mathrm{ind}(T_n) = \mathrm{ind}(T_m)$ for every $m \geq n$. Therefore it makes sense to define the index of a B-Fredholm operator as the index of the Fredholm operator T_n.

For an Atkinson-type characterization, we need a generalization of the notion of invertibility, namely the Drazin inverse.

Let A be a unital Banach algebra. We say that an element $a \in A$ is *Drazin invertible* if there is some $b \in A$ and $k \in \mathbb{N}$ such that

$$a^k = a^k ba, \qquad b = bab, \qquad ba = ab. \tag{6.1}$$

Such b is unique if it exists, and we will call it the Drazin inverse of a. The least k for which (6.1) holds is called the *Drazin index* of a. The set of all Drazin invertible elements of A will be denoted by $\mathcal{D}(A)$.

Note that if $b \in A$ is a Drazin inverse for $a \in A$ of degree less or equal than 1, then a is group invertible, and b is the group inverse for a. In a semisimple Banach algebra we have $\mathrm{soc}(A) \subseteq \overset{\sqcap}{A} \subseteq \mathcal{D}(A)$.

Theorem 6.3.1 (7, Theorem 3.4) *(Atkinson-type characterization) Let $T \in B(X)$. Then T is a B-Fredholm operator if and only if $\pi_0(T)$ is Drazin invertible in the algebra $B(X)/F_0(X)$.*

We should note that in Atkinson's theorem for Fredholm operators we could use either $K(X)$ or $F_0(X)$, but for B-Fredholm we cannot. The Fredholm operators are stable under compact perturbations, but B-Fredholm are not: take the zero operator O, which is B-Fredholm, and an operator $K \in K(X)$ such that $R(K)$ is not finite, then $O+K = K$ is not a B-Fredholm operator since $R(K)$ is not closed.

C. Schmoeger proved that an operator T is generalized Fredholm if and only if $\pi(T)$ is group invertible in $B(X)/F_0(X)$ ([19, Theorem 3.3]). Hence a generalized Fredholm operator is a B-Fredholm operator, but the converse is not true, as we have that a nilpotent operator with a non-closed range is B-Fredholm but not generalized Fredholm since it is not relatively regular.

There is an interesting connection between B-Fredholm and generalized Fredholm operators.

Theorem 6.3.2 (7, Proposition 3.3) *Let $T \in B(X)$. Then T is a B-Fredholm operator if and only if there exists a positive integer k such that T^k is a generalized Fredholm operator.*

From Theorem 6.3.1 we know that B-Fredholm operators are precisely those whose image under the natural homomorphism π_0 is Drazin invertible. When dealing with arbitrary Banach algebras, this Atktinson-type characterization will be our main motivation. As we are using the Drazin invertibility, we shall employ the term "Drazin-Fredholm" in order to emphasize the role played by the Drazin invertible elements in the quotient algebra. Thus, the B-Fredholm operators are in this sense the "Drazin-Fredholm" elements of the algebra $B(X)$. Now we proceed to define the Drazin-Fredholm elements in a semi-simple algebra:

Definition 6.3.2 *Let A be a semisimple algebra, then $x \in A$ is a Drazin-Fredholm element if $\pi(x)$ is Drazin invertible in $A/\mathrm{soc}(A)$.*

We have already seen in the comments preceding Theorem 6.3.2 an e-xample of a Drazin-Fredholm element of $B(X)$ which is not generalized Fredholm; now some examples of Drazin-Fredholm elements:

Example 6.3.1 *(1) Nilpotent elements are Drazin-Fredholm.*

Let $a \in A$ be such that $a^n = 0$ for some $n \in \mathbb{N}$. Then $\pi(a)^n = \pi(a^n) = \pi(0)$, so $\pi(a)$ is Drazin invertible and a is Drazin-Fredholm.

(2) Generalized Fredholm elements are Drazin-Fredholm elements.

If $a \in \Phi_g(A)$, then $\pi(a)$ is group invertible, i.e., it is Drazin invertible of degree 0 or 1. Thus a is Drazin-Fredholm.

Proposition 6.3.1 *Let $a, b \in A$ be Drazin-Fredholm elements.*
(1) If $ab \in \mathrm{soc}(A)$ and $ba \in \mathrm{soc}(A)$ then $a + b$ is a Drazin-Fredholm element.
(2) If $ab = ba$ then ab is a Drazin-Fredholm element.
(3) If $s \in \mathrm{soc}(A)$ then $a + s$ is a Drazin-Fredholm element.

Proof. (1) Since $ab, ba \in \mathrm{soc}(A)$, then $\pi(ab) = \pi(ba) = 0$, and by [12, Corollary 1] we have $\pi(a) + \pi(b)$ is Drazin invertible, therefore $a + b$ is a Drazin-Fredholm element.

(2) Since $ab = ba$, we have $\pi(ab) = \pi(ba)$, and by [7, Proposition 2.6] we have $\pi(ab)$ is Drazin invertible, therefore ab is a Drazin-Fredholm element.

(3) If $s \in \mathrm{soc}(A)$ then $\pi(a+s) = \pi(a) + \pi(s) = \pi(a)$ is Drazin invertible, therefore $a + s$ is a Drazin-Fredholm element.□

Now we proceed to the case where A is an arbitrary Banach algebra.

Let A be an algebra. Given an ideal J of A we say that $a \in A$ is a Fredholm element relative to J if $[a]$ is invertible in A/J, where $[a]$ stands for the coset $a + J$. Following the operator case, we say a is a generalized Fredholm element relative to J if $[a]$ is relatively regular and for some generalized inverse $[b]$ of $[a]$ we have $[e - ab - ba] \in (A/J)^{-1}$. Finally, an element $a \in A$ is Drazin-Fredholm element relative to J if $[a]$ is Drazin invertible in A/J.

Using the properties of the Drazin inverse, we get at once the following results:

Theorem 6.3.3 *The element $a \in A$ is a Drazin-Fredholm element if and only if, for some positive integer p, a^p is a generalized Fredholm element of A.*

Proof. It follows from [18, Lemma 2] by noting that if b is the group inverse of a^k then $ab = ba$.□

Proposition 6.3.2 *The following statements are equivalent:*
(a) a is a Drazin-Fredholm element.
(b) a^n is a Drazin-Fredholm element for every $n \in \mathbb{N}$.
(c) There is n such that a^n is a Drazin-Fredholm element.

Proof. (a) $\Rightarrow$ (b) [12, Theorem 2], (b) $\Rightarrow$ (c) is clear, (c) $\Rightarrow$ (a) follows from Theorem 6.3.3.□

Note that in the operator case, we used the ideal $F_0(X)$, and every element of $F_0(X)$ is relatively regular since a finite rank operator has closed and complemented range and null space. Following this, an ideal J is called *completely regular* if $J \subseteq \overset{\sqcap}{A}$.

If A is a semisimple algebra, it is known that $\mathrm{soc}(A) \subseteq \overset{\sqcap}{A}$. Also if J is an ideal of A such that $J \subseteq \overset{\sqcap}{A}$ then $J \subseteq \mathrm{soc}(A)$ (see [9, Theorem 7]). Thus, if $\pi : A \to A/\mathrm{soc}(A)$ is the natural homomorphism, then an element $a \in A$ is a Drazin-Fredholm element if $\pi(a)$ is Drazin invertible in $(A/\mathrm{soc}(A))$.

6.4 T-Fredholm elements

Until now we have used the natural homomorphism π, we now ask if we can drop this restriction. This idea was investigated by R. Harte [15], who defined the T-Fredholm elements using an arbitrary homomorphism T in the following way:

Definition 6.4.1 (15) *Let A, B unital algebras and $T : A \to B$ a homomorphism. An element $a \in A$ is T-Fredholm if $T(a)$ is invertible in B.*

Recall that for a homomorphism T, we have $T(ab) = T(a)T(b)$ and $T(e) = e$. It is clear that $T(A^{-1}) \subset B^{-1}$.

Example 6.4.1 *(1) Let $A = C(X)$ and $B = C(Y)$ be the Banach spaces of continuous functions on compact Hausdorff spaces X and Y. If $T : A \to B$ is induced by composition with a continuous map $\varphi : Y \to X$, then $a \in A$ is Fredholm if and only if its null set $a^{-1}(0)$ is disjoint from the image $\varphi(Y)$ of Y in X.*

(2) Let $A = B(X)$ and $B = A/J$, where J is any closed ideal between the finite rank and compact operators. Then, by Atkinson's theorem, the Fredholm elements are the usual operators with finite-dimensional null space and closed range of finite codimension.

Motivated by the definition of Caradus ([10]) on generalized Fredholm operators we will now define the generalized T-Fredholm elements.

Definition 6.4.2 *[16] Let A, B be unital algebras and $T : A \to B$ a homomorphism. We say that $a \in A$ is a generalized T-Fredholm element if $T(a) \in \widehat{B}$ and for some generalized inverse a' of $T(a)$ holds*

$$e - T(a)a' - a'T(a) \in B^{-1}.$$

We will denote by $\Phi_g^T(A)$ the set of all generalized T-Fredholm elements.

When $B = A$ and $T = I$ then $\Phi_g^T(A) = A^g$ coincides with the "generalized invertibles" of Schmoeger ([19]):

$$A^g = \{a \in A : \exists b \in A,\ a = aba,\ e - ab - ba \in A^{-1}\}.$$

Proposition 6.4.1 *([16], Proposition 2.2) Suppose $T(A) = B$ and $T^{-1}(0) \subseteq \overset{\sqcap}{A}$. Then $T^{-1}(\overset{\sqcap}{B}) \subseteq \overset{\sqcap}{A}$. Moreover, if a is a T-Fredholm element, then a is relatively regular and for some generalized inverse b holds $e - ab - ba \in T^{-1}(B^{-1})$.*

From this proposition we see that if T is surjective and finitely regular then $\Phi^T(A) \subseteq \Phi_g^T(A)$. On the other hand, if $p \in A$ is an idempotent, then $pep = p^2 = p$ and $T(pep) = T(p)$ so $T(p)$ is relatively regular in B, and since $(e - ep - pe)^2 = (e - 2p)^2 = e$ we have that $e - ep - pe \in B^{-1}$ and $p \in \Phi_g^T(A)$. Thus the idempotents are generalized T-Fredholm elements, but since idempotent $T(p)$, $p \neq e$, is not invertible in the algebra B, p is not T-Fredholm element. Hence, the inclusion $\Phi^T(A) \subseteq \Phi_g^T(A)$ can be proper.

Thus, it makes sense to consider the restrictions $T(A) = B$ and $T^{-1}(0) \subseteq \overset{\sqcap}{A}$. If A is a semisimple Banach algebra and T is finitely regular and onto, then $T^{-1}(0)$ is a subset of the socle ([9]).

In a unital Banach algebra, the generalized invertibles and the group invertibles coincide ([16]). For the generalized T-Fredholm elements we have a characterization analogous to the case of generalized Fredholm elements:

Theorem 6.4.1 *([16], Theorem 2.4) Let $a \in A$. Then $a \in \Phi_g^T(A)$ if and only if $T(a) \in B^g$.*

And now some properties found in [16].

Proposition 6.4.2 *Let $a, b \in A$.*
(1) If $a, b \in \Phi_g^T(A)$ and $ab = ba$, then $ab \in \Phi_g^T(A)$.
(2) If $a \in \Phi_g^T(A)$, then $a^n \in \Phi_g^T(A)$ for every $n \in \mathbb{N}$.
(3) $e - ab \in \Phi_g^T(A)$ if and only if $e - ba \in \Phi_g^T(A)$.

Proof- (1) ([16], Proposition 2.6), (2) ([16], Corollary 2.7), (3) ([16], Proposition 2.8(2)).□

Following the same idea as in the preceding section we may replace the usual invertibility with Drazin invertibility to define the T-Drazin Fredholm elements:

Definition 6.4.3 *Let A, B unital algebras and $T : A \to B$ a homomorphism. An element $a \in A$ is T-Drazin Fredholm if $T(a)$ is Drazin invertible in B.*

The set of all T-Drazin Fredholm elements of A will be denoted by $\Phi_D^T(A)$.

The following properties of T-Drazin Fredholm elements follows from the algebraic properties of the Drazin inverse.

Proposition 6.4.3 *Let* $a, b \in \Phi_D^T(A)$. *If* $ab = ba$ *then* $ab \in \Phi_D^T(A)$.

Proof- It follows from the first part of [7, Proposition 2.6].□

Proposition 6.4.4 *If* $a, b \in \Phi_D^T(A)$ *and* $ba, ab \in N(T)$, *then* $a+b \in \Phi_D^T(A)$.

Proof. It follows from [12, Corollary 1].□

Example 6.4.2 *Let* H *be a (non-separable) Hilbert space with dimension* $h > \aleph_0$. *Let* $\mathcal{I}_\alpha$ *denote the two-sided ideal in* $B(H)$ *of all bounded linear operators of rank less than* α, $1 \leq \alpha \leq h$ *and* $\mathcal{T}_\alpha = \overline{\mathcal{I}_\alpha}$ *Notethat if we let* $\alpha = \aleph_0$, *then* $\mathcal{T}_\alpha = K(H)$.

Let $A = B(H)$, $B = B(H)/\mathcal{T}_\alpha$, $1 \leq \alpha \leq h$ *and* $T = \pi : A \to B$ *the natural homomorphism. This situation was studied in [13].*

A subspace M *of a Hilbert space* H *is called* α*-closed if there is a closed subspace* N *of* H *such that* $N \subset M$ *and such that* $\dim(K \cap M^\perp) < \alpha$, *where* $M^\perp$ *denotes the orthogonal complement of* M.

An operator $A \in B(H)$ *is called an* α*-Fredholm operator if* $\dim N(T) \leq \alpha$, $\dim H/R(T) \leq \alpha$ *and the range of* A *is* α*-closed.*

Theorem 6.4.2 *(Atkinson-type characterization) For* $A \in B(H)$ *the following assertions are equivalent:*
(a) A *is* α*-Fredholm;*
(b) $T(A)$ *is invertible in* $B = B(H)/\mathcal{T}_\alpha$*;*
(c) A *is invertible modulo* $\mathcal{I}_\alpha$.

We can define the B_α-Fredholm operators in the following way: An operator $A \in B(H)$ is called B_α-Fredholm operator if $T(A)$ is Drazin invertible in the quotient algebra $B(H)/\mathcal{T}_\alpha$.

It is worth noticing here that if a is Drazin invertible with Drazin inverse b then $a(e-ab)$ is nilpotent. J. Koliha has introduced in [17] a generalization of Drazin invertible elements by considering $a(e-ab)$ not nilpotent but quasinilpotent.

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7

Planteamiento Operacional del Problema de Cauchy mediante Potenciales de Superficie

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Resumen: En este trabajo realizamos el planteamiento operacional del problema de Cauchy para la ecuación de Laplace en una región anular en el plano. Para esto se realiza el análisis de la solubilidad de un problema auxiliar y se demuestra la existencia y unicidad de las soluciones clásica y débil así como su representación como suma de potenciales de superficie. Esto último permite proponer un algoritmo para hallar la solución del problema auxiliar, a través de un sistema de ecuaciones integrales. Este algoritmo se desarrolla para el caso de una región anular circular y se valida por medio de ejemplos sintéticos. Con esta misma técnica es posible desarrollar un algoritmo estable para el problema de Cauchy.

7.1 Introducción

El problema que se estudia en este trabajo es el de determinar una función armónica sobre una región anular en el plano, a partir de la información parcial (datos de Cauchy) que se tiene de dicha función en una parte de la frontera de dicha región anular.

El problema anterior es de gran importancia en las aplicaciones, ya que por medio de su solución se puede, por ejemplo, determinar el deterioro de la pared interna de una tubería, conocer el potencial en una región a la que no se tiene acceso, determinar el daño que tiene una lámina (conocidos como cracks), entre otros ([1], [9]). Otras aplicaciones son la Tomografía de capacitancias y la Tomografía Médica ([3]), [5]).

Sea Ω una región anular en $\mathbb{R}^2$. Por $S = S_1 \cup S_2$ denotamos su frontera, donde S_1 es la frontera interior y S_2 es la frontera exterior.

Consideremos el problema siguiente: dada una función conocida v definida sobre S_2, con ciertas propiedades que precisaremos posteriormente, encontrar una función u definida en Ω que satisfaga el problema de contorno:

$$\begin{aligned} \Delta u = 0 &\quad \text{en} \quad \Omega, \\ u = v &\quad \text{en} \quad S_2, \\ \frac{\partial u}{\partial n} = 0 &\quad \text{en} \quad S_2. \end{aligned} \tag{7.1}$$

Este problema se encuentra dentro de la clase conocida como problemas de tipo Cauchy y su solución conduce a un problema mal planteado debido a la sensibilidad de la misma con respecto a errores en v. Una manera equivalente de formular este problema consiste en recuperar el potencial u, solución de (1), en S_1, a partir del conocimiento de $u = v$ en S_2.

Para obtener un planteamiento operacional que permita el análisis del problema en una segunda formulación, usaremos un problema auxiliar que se presentará en la sección siguiente. Este problema, mediante las técnicas de la teoría de potencial, será llevado a un sistema de ecuaciones integrales equivalente en el que se deben determinar densidades de carga definidas sobre la frontera de la región. También se demuestra que las soluciónes clásica y débil pueden buscarse como la suma de un potencial de capa doble más un potencial de capa simple, lo que permite proponer un algoritmo para identificar las densidades y con ello el potencial u, solución del problema de contorno (7.1).

Para el caso de una región anular circular se implementa este algoritmo para encontrar una solución aproximada del problema auxiliar, el cual puede extenderse a regiones más generales; esto último no será desarrollado en este trabajo. Este algoritmo se reduce a encontrar la solución de un sistema de ecuaciones lineales algebraicas obtenido por un método de colocación. Se validan el algoritmo y los programas elaborados comparando las soluciones aproximadas obtenidas por el algoritmo con las respectivas soluciones exactas.

7.2 Solubilidad del problema auxiliar

Para el análisis del problema de Cauchy, se utiliza el siguiente problema auxiliar:

$$
\begin{aligned}
\Delta u &= 0 \quad \text{en} \quad \Omega, \\
u &= \varphi \quad \text{en} \quad S_1, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{en} \quad S_2,
\end{aligned}
\tag{7.2}
$$

el cual consiste en determinar u si conocemos φ en S_1. Este problema es bien planteado ([1]). La relación de este problema con el problema (7.1) será dada con detalle en la tercer sección de este artículo.

7.2.1 Solución clásica y solución débil

Consideremos los siguientes espacios de funciones: $L_2(\Omega)$ y $L_2(S_i)$, los espacios de funciones de cuadrado integrable definidas sobre Ω y S_i, $i = 1, 2$; $H^1(\Omega)$ el espacio de Sobolev de funciones cuya primera derivada generalizada pertenece a $L_2(\Omega)$. Con $H^{1/2}(S_i)$, $i = 1, 2$, denotamos al espacio de las funciones de $L_2(S_i)$ que son traza ([10], [7]) a S_i, $i = 1, 2$, de alguna función de $H^1(\Omega)$.

Definimos los dos sentidos en los cuales vamos a entender la solución.

Definición 7.2.1 *La función u es solución clásica del problema de contorno (2) si, u es una función que pertenece a $C^2(\Omega) \cap C(\Omega \cup S_1) \cap C^1(\Omega \cup S_2)$ y satisface las condiciones del problema en el sentido usual.*

Definición 7.2.2 *Dada $\varphi \in H^{\frac{1}{2}}(S_1)$, la función $u \in H^1(\Omega)$ es solución débil del problema de contorno (2) si $u|_{S_1} = \varphi$ y se cumple*

$$\int_\Omega \nabla u \cdot \nabla v d\Omega = 0, \tag{7.3}$$

para cada $v \in H_0^1(\Omega)$.

7.2.2 Existencia de las soluciones

Teorema 7.2.1 *Para cada $\varphi \in H^{\frac{1}{2}}(S_1)$ la solución débil u del problema (2) existe y es única.*

Demostración: Por ser $\varphi \in H^{\frac{1}{2}}(S_1)$ existe $\Phi \in H^1(\Omega)$ tal que $-\Phi|_{S_1} = \varphi$. Por ello la solución débil de (2) se puede expresar en la forma $\overline{u} = u - \Phi$ con $\overline{u} \in H_{0,1}^1(\Omega) = \left\{ w \in H^1(\Omega) : w|_{S_1} = 0 \right\}$. Entonces el problema (2) es equivalente a encontrar $\overline{u} \in H_0^1(\Omega)$ tal que

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \nabla \overline{u}\ \nabla v dx = \int_\Omega \nabla \Phi \nabla v dx. \tag{7.4}$$

El término en la parte derecha de la igualdad (7.4) es un funcional lineal y continuo en $H_0^1(\Omega)$ pues

$$\left|\int_\Omega \nabla\Phi\nabla v dx\right| \leq \|\nabla\Phi\|_{L_2(\Omega)} \|\nabla v\|_{L_2(\Omega)}$$

y $\|\nabla v\|_{L_2(\Omega)}$ es una norma equivalente a la norma usual en $H_{0,1}^1(\Omega)$. Según el teorema de representación de Riesz, existe entonces una única función $\overline{u} \in H_0^1(\Omega)$ que satisface (7.4).$\square$

Teorema 7.2.2 *Para cada $\varphi \in C(S_1)$ la solución clásica del problema (2) existe y se puede expresar como la suma de un potencial de capa doble más un potencial de capa simple.*

Demostración : Sean ρ_1 y ρ_2 densidades definidas sobre $C(S_1)$ y $C(S_2)$ respectivamente. Con estas densidades definimos los potenciales de capa doble y capa simple

$$\int_{S_1} \rho_1(y) \frac{\partial}{\partial n_y} \Phi(x,y) ds_y, \quad \int_{S_2} \rho_2(y)\Phi(x,y)ds_y,$$

donde $\Phi(x,y)$ es la solución fundamental de la ecuación de Laplace para $\mathbb{R}^2$ la cual está dada por

$$\Phi(x,y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|}. \tag{7.5}$$

Buscamos la solución u del problema (7.2) en la forma

$$u(x) = \int_{S_1} \rho_1(y) \frac{\partial}{\partial n_y} \Phi(x,y) ds_y + \int_{S_2} \rho_2(y)\Phi(x,y)ds_y \quad x \in \Omega, \tag{7.6}$$

es decir, como una suma de un potencial de capa doble más un potencial de capa simple.

Como ρ_1 y ρ_2 son continuas, los potenciales que definen estas densidades son funciones armónicas sobre Ω ([8]), es decir, u satisface la ecuación de Laplace. Para que las condiciones de contorno sobre S_1 y S_2 se cumplan debemos tener

$$u(x) = \varphi(x) \quad \text{en} \quad S_1,$$

$$\frac{\partial u}{\partial n_x} = 0 \quad \text{en} \quad S_2.$$

Para la condición de contorno sobre S_1 tenemos

$$\begin{aligned} P.V. \int_{S_1} \rho_1(y) \frac{\partial}{\partial n_y} \Phi(x,y) ds_y + \frac{1}{2}\rho_1(x) + \int_{S_2} \rho_2(y)\Phi(x,y) ds_y \\ = \varphi(x), \end{aligned} \tag{7.7}$$

para cada $x \in S_1$ y para la condición de contorno sobre S_2 tenemos

$$\begin{aligned} \int_{S_1} \rho_1(y) \frac{\partial}{\partial n_x} \left(\frac{\partial}{\partial n_y} \Phi(x,y) \right) ds_y + P.V. \int_{S_2} \rho_2(y) \frac{\partial}{\partial n_x} \Phi(x,y) ds_y \\ + \frac{1}{2}\rho_2(x) = 0, \end{aligned} \tag{7.8}$$

para cada $x \in S_2$, donde P.V. denota el valor principal de Cauchy de la integral [6]. Para hallar el sistema (7.7)-(7.8) se han utilizado las fórmulas de salto de Sojotvsky [8].

Definimos ahora los operadores

$$K_{ij} : C(S_j) \longrightarrow C(S_i),\ i,j = 1,2,$$

donde:

$$K_{11}(\mu)(x) = P.V. \int_{S_1} \mu(y) \frac{\partial}{\partial n_y} \Phi(x,y) ds_y, \tag{7.9}$$

$$K_{12}(\mu)(x) = \int_{S_2} \mu(y) \Phi(x,y) ds_y, \tag{7.10}$$

$$K_{21}(\mu)(x) = \int_{S_1} \mu(y) \frac{\partial}{\partial n_x} \left(\frac{\partial}{\partial n_y} \Phi(x,y) \right) ds_y, \tag{7.11}$$

$$K_{22}(\mu)(x) = P.V. \int_{S_2} \mu(y) \frac{\partial}{\partial n_x} \Phi(x, y) ds_y, \tag{7.12}$$

Si utilizamos la notación:

$$\rho = [\rho_1,\ \rho_2]^t, \quad \overline{\varphi} = [\varphi,\ 0]^t, \quad I = [I_1,\ I_2]^t,$$

donde I_1 y I_2 denotan el operador identidad que actúa en los espacios $C(S_1)$ y $C(S_2)$ respectivamente, y definimos la matriz K como

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix},$$

el sistema de ecuaciones integrales (7.7)-(7.8) se lleva a la ecuación operacional

$$\left(K + \frac{1}{2} I \right) \rho = \overline{\varphi}. \tag{7.13}$$

Así, el problema de existencia de la solución clásica se ha reducido a analizar la solubilidad de la ecuación operacional (7.13), que es una ecuación de Fredholm de segunda especie, tiene solución. El operador K que actúa sobre $C(S_1) \times C(S_2)$ con la norma $\| (\rho_1, \rho_2) \| = \max\{\| \rho_1 \|_\infty, \| \rho_2 \|_\infty\}$ es compacto (cada uno de los operadores $K_{ij},\ i, j = 1, 2$, es compacto) y se puede ver, utilizando la unicidad de solución del problema (1) siguiendo [4], que $\lambda = -\frac{1}{2}$ no es un valor propio de este operador. Como el espacio de las funciones continuas no es un espacio de Hilbert, no se puede aplicar la alternativa de Fredholm; pero se pueden aplicar los teoremas de Riesz ([8]) de donde se obtiene que $K + I$ es inyectivo y sobreyectivo, así que su inverso $\left(K + \frac{1}{2} I\right)^{-1}$ existe y es continuo. De este modo, dada $\overline{\varphi}$ encontramos $\rho = [\rho_1,\ \rho_2]^t$ como

$$\rho = \left(K + \frac{1}{2} I \right)^{-1} (\overline{\varphi}). \tag{7.14}$$

Por lo tanto, la solución clásica del problema (2) existe y está dada por (7.6).
□

Lo anterior muestra que el problema 2 es equivalente al sistema lineal de ecuaciones integrales (7.7)-(7.8), así que el estudio de la solubilidad de (2) puede hacerse a través del mencionado sistema.

La demostración del teorema (7.2.2) da un algoritmo para encontrar la solución clásica del problema (2).

7.3 Planteamiento operacional y existencia global

Se realiza el planteamiento operacional de la manera siguiente: supongamos que se cumplen las condiciones necesarias para la existencia de la solución de los problemas (7.1) y (7.2) y sea $A : H^{\frac{1}{2}}(S_1) \to H^{\frac{1}{2}}(S_2)$ el operador que asocia a cada φ la traza a S_2 de la solución débil u del problema de contorno (7.2). Es decir, la regla de correspondencia para el operador A es

$$A(\varphi) = tr(u). \tag{7.15}$$

donde el lado derecho de (15) es la traza de u a S_2.

Debemos notar que la ecuación (7.15) que nos da la relación entre el problema de Cauchy y el problema auxiliar no tiene solución para cualquier función $v \in L_2(S_2)$. Esto quiere decir que si buscamos la solución clásica u del problema de Cauchy obtenida a partir de v, entonces se deben imponer condiciones adicionales a la función v. Es importante notar que el operador $A : H^{\frac{1}{2}}(S_1) \to L_2(S_2)$ es compacto e inyectivo y de ahí el mal planteamiento del problema de Cauchy ([1]).

El problema de contorno auxiliar dado por (7.2) está relacionado con el problema de Cauchy de la manera siguiente:

La solución del problema (7.2) *es también solución del problema* (7.1) *si elegimos la condición de contorno* φ *de forma que satisfaga l a ecuación operacional* $A(\varphi) = tr(u) = v$*, donde* u *es la solución del problema* (7.2) *correspondiente a la condición de contorno* $u|_{S_1} = \varphi$ *y* v *está dada en el planteamiento del problema* (7.1)*, es decir, si elegimos* φ *como* $\varphi = A^{-1}(v)$*.*

Daremos las condiciones de existencia global de la solución del problema (1) en el caso en que Ω consta de una región anular circular. La solución u del problema de Cauchy (1) puede buscarse por medio de armónicos circulares [3]. Se ha visto en [2] que si la condición de contorno v se expresa como

$$v(\theta) = \frac{v_0}{2} + \sum_{k=1}^{\infty} v_k^1 \cos k\theta + \sum_{k=1}^{\infty} v_k^2 \sin k\theta,$$

donde v_k^1 y v_k^2 son los coeficientes de Fourier de v, entonces, la expresión correspondiente para la solución u es

$$u(r,\theta) = \frac{v_0}{2} + \sum_{k=1}^{\infty} \left(\frac{r^k}{2R_2^k} + \frac{R_2^k}{2r^k} \right) \left(v_k^1 \cos k\theta + v_k^2 \sin k\theta \right),$$

y además que la ecuación operacional (7.15) tiene solución $\varphi \in L_2(S_1)$ para cada $v \in L_2(S_1)$ cuyos coeficientes de Fourier v_k^1 y v_k^2 satisfagan

$$\sum_{k=1}^{\infty} \left(\frac{R_2}{R_1} \right)^{2k} \left[(v_k^1)^2 + (v_k^2)^2 \right] < \infty.$$

De acuerdo con [10] y [12], para que la traza a S_1 de la solución del problema de Cauchy pertenezca a $H^{\frac{1}{2}}(S_1)$ se debe tener

$$\sum_{k=1}^{\infty} k \left(\frac{R_2}{R_1} \right)^{2k} \left[(v_k^1)^2 + (v_k^2)^2 \right] < \infty.$$

Para la derivada normal de u en S_1 dada por

$$\frac{1}{2R_1} \sum_{k=1}^{\infty} k \left[\left(\frac{R_1}{R_2} \right)^k - \left(\frac{R_2}{R_1} \right)^k \right] \left(v_k^1 \cos k\theta + v_k^2 \sin k\theta \right),$$

se tiene que esta pertenece a $L_2(S_1)$ si se cumple la condición adicional [12]:

$$\sum_{k=1}^{\infty} k^2 \left(\frac{R_2}{R_1} \right)^{2k} \left(|v_k^1|^2 + |v_k^2|^2 \right) < \infty.$$

Bajo la condición anterior, que deben satisfacer los coeficientes de Fourier de v, se garantiza que las soluciones clásica y débil existen y coinciden.

7.4 Búsqueda de la solución débil

Para poder buscar la solución débil del problema (2) por medio de la ecuación operacional (7.13), es necesario extender cada uno de los operadores K_{ij}, $i, j = 1, 2$, que conforman la matriz K a los espacios $L_2(S_j)$, $j = 1, 2$, pues la solución débil debe pertenecer al espacio $H^1(\Omega) \subset L_2(\Omega)$. Los operadores K_{ij} pueden extenderse de $C(S_j)$ a $L_2(S_j)$, $j = 1, 2$ pues estas extensiones son operadores compactos [11].

La definición siguiente, basada en los trabajos realizados en [4], nos ayuda a demostrar que la solución débil puede buscarse por medio de la ecuación operacional (7.13).

Definición 7.4.1 *Dada $\varphi \in L_2(S_1)$ diremos que el problema auxiliar*

$$\begin{aligned} \Delta u &= 0 \ \text{ en } \Omega, \\ u &= \varphi \ \text{ en } S_1, \\ \frac{\partial u}{\partial n} &= 0 \ \text{ en } S_2, \end{aligned}$$

es soluble en sentido de la teoría de potencial, si existe una sucesión u_n de soluciones clásicas de los problemas

$$\begin{aligned} \Delta u_n &= 0 \ \text{ en } \Omega, \\ u_n &= \varphi_n \ \text{ en } S_1, \\ \frac{\partial u_n}{\partial n} &= 0 \ \text{ en } S_2, \end{aligned}$$

donde $\varphi_n \longrightarrow \varphi$ en $L_2(S_1)$. Si además existe el límite de u_n en $L_2(\Omega)$ diremos que dicho límite es solución en sentido de la teoría de potencial del problema auxiliar.

Claramente se cumple que el problema (2) es soluble en sentido de teoría de potencial para cualquier $\varphi \in L_2(S_1)$.

Proposition 7.4.1 *Para cada $\varphi \in H^{\frac{1}{2}}(S_1)$ la solución en sentido de teoría de potencial del problema (2) existe y coincide con la solución débil.*

Demostración Sea $\varphi \in H^{\frac{1}{2}}(S_1)$, entonces existe una sucesión $\varphi_n \in C^1(S_1)$ tal que $\varphi_n \longrightarrow \varphi$ en $L_2(S_1)$. La soluciones clásica y débil coinciden en este caso para cada φ_n [10]. Denotemos por u_n la solución correspondiente a φ_n. Como las soluciones u_n son armónicas y se extienden hasta la frontera, son acotadas. Como cada φ_n también es continua, podemos expresar cada una de las soluciones clásicas u_n como la suma de un potencial de capa doble más un potencial de capa simple. Ahora, comprobaremos que la sucesión de soluciones u_n es convergente en $L_2(\Omega)$.

Sean

$$u_n(x) = \int_{S_1} \rho_1^n(y) \frac{\partial}{\partial n_y} \Phi(x,y) ds_y + \int_{S_2} \rho_2^n(y) \Phi(x,y) ds_y$$

y

$$u_m(x) = \int_{S_1} \rho_1^m(y) \frac{\partial}{\partial n_y} \Phi(x,y) ds_y + \int_{S_2} \rho_2^m(y) \Phi(x,y) ds_y.$$

Tenemos entonces que

$$\begin{aligned} \|u_n - u_m\| &= \int_{S_1} (\rho_1^n - \rho_1^m)(y) \frac{\partial}{\partial n_y} \Phi(x,y) ds_y + \int_{S_2} (\rho_2^n - \rho_2^m)(y) \Phi(x,y) ds_y \\ &\leq M \left\| \rho^n - \rho^m \right\|, \end{aligned}$$

donde

$$\rho^n - \rho^m = \begin{bmatrix} \rho_1^n - \rho_1^m \\ \rho_2^n - \rho_2^m \end{bmatrix}.$$

En este punto podemos pensar en $\rho^n - \rho^m$ como el par de densidades que se buscan para encontrar la solución del problema

$$\begin{aligned} \Delta u = 0 \quad &\text{en} \quad \Omega, \\ u = \varphi_n - \varphi_m \quad &\text{en } S_1, \\ \frac{\partial u}{\partial n} = 0 \quad &\text{en } S_2. \end{aligned}$$

De lo que hemos visto anteriormente, estas densidades se pueden encontrar como

$$[\rho^n - \rho^m]^t = \left[K + \frac{1}{2}I\right]^{-1} [(\varphi_n - \varphi_m), \quad 0]^t$$

donde $K + \frac{1}{2}I$ es el operador dado en (7.13) con inverso continuo y como φ_n es una sucesión convergente se tiene que:

$$[(\varphi_n - \varphi_m), \quad 0]^t \longrightarrow 0 \quad \text{y} \quad \left[K + \frac{1}{2}I\right]^{-1} [(\varphi_n - \varphi_m), \quad 0]^t \longrightarrow 0.$$

Por lo tanto $\|u_n - u_m\| \longrightarrow 0$, así u_n es una sucesión de Cauchy y como es una sucesión de funciones armónicas en $L_2(\Omega)$ (pues cada solución clásica u_n coincide con la solución débil, de aquí que $u_n \in H^1(\Omega) \subset L_2(\Omega)$), converge en $L_2(\Omega)$ a una función u que es armónica. De aquí u satisface la condición integral en la definición de solución débil (definición 10). Por otra parte, ya que el operador traza es acotado y $u_n \longrightarrow u$ entonces $tr(u_n) = \varphi_n \longrightarrow \varphi = tr(u)$, donde $tr(u)$ es la traza de u a S_1. $\square$

En la sección siguiente consideramos el caso en que Ω es una región anular circular y desarrollamos el sistema de ecuaciones integrales (13) para ver su factibilidad en la solución del problema auxiliar (2). Ya que muchas regiones anulares pueden transformarse en una región anular circular mediante transformaciones conformes conocidas, este sistema permitiría dar solución en estos casos. Sin embrago, en otrosd es muy complicado hallar dicha transformación conforme y debemos aplicar el sistema (13) en la región original.

7.5 Caso de una región anular circular

Supongamos que Ω es una región anular circular con frontera $S = S_1 \cup S_2$, es decir, Ω consiste de dos círculos concéntricos de radios R_1 y R_2 con $R_1 < R_2$. Vamos a resolver en este caso el sistema de ecuaciones integrales (7.7)-(7.8). Después de realizar los cálculos, el sistema este se reduce a

$$\begin{aligned} k_1 \int_{S_1} \rho_1(y) ds_y + \rho_1(x) + \frac{1}{\pi} \int_{S_2} \rho_2(y) \ln\left(\frac{1}{|x-y|}\right) ds_y \\ = 2\varphi(x) \quad x \in S_1, \end{aligned} \tag{7.16}$$

$$k_2 \int_{S_1} \frac{\rho_1(y)}{|x-y|^4} ds_y + k_3 \int_{S_1} \frac{\rho_1(y)}{|x-y|^2} ds_y + k_4 \int_{S_2} \rho_2(y) \\ + \rho_2(x) = 0 \quad x \in S_2, \tag{7.17}$$

donde

$$k_1 = -\frac{1}{2\pi R_1}, \ k_2 = \frac{1}{\pi}\left(R_1 - \frac{R_2^2}{R_1}\right)\left(\frac{R_2}{2} - \frac{R_1^2}{2R_2}\right), k_3 = -\frac{1}{2\pi R_2}\left(\frac{R_2^2}{R_1} - R_1\right),$$

$$k_4 = -\frac{1}{2\pi R_2}.$$

La ecuación (7.16) corresponde a la condición de contorno sobre S_1 y la ecuación (7.17) corresponde a la condición de contorno sobre S_2. En este sistema no hay problemas de singularidad debido a lo siguiente: la ecuación (7.17) se cumple para cada $x \in S_2$ y la integración es sobre S_1, la ecuación (7.16) se cumple para cada $x \in S_1$ y la segunda integral en esta ecuación se realiza sobre S_2, así que en ambas ecuaciones tenemos $x - y \neq 0$.

Tomando en cuenta la geometría de la región Ω podemos buscar ρ_1 y ρ_2 en su desarrollo en serie de Fourier si estas densidades son suficientemente suaves. En lo que sigue asumiremos que esta condición se satisface. Supongamos entonces que

$$\rho_1(\theta) = \frac{a_0^1}{2} + \sum_{k=1}^{\infty} a_k^1 \cos k\theta + \sum_{k=1}^{\infty} b_k^1 \sin k\theta, \tag{7.18}$$

$$\rho_2(\theta) = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2 \cos k\theta + \sum_{k=1}^{\infty} b_k^2 \sin k\theta. \tag{7.19}$$

Sustituyendo en el sistema de ecuaciones integrales (7.16)-(7.17) obtenemos

$$-\frac{a_0^1}{2} + \rho_1(\overline{\theta}) + \frac{1}{\pi} \int_{S_2} \rho_2(\theta) \ln f(\theta, \overline{\theta}) d\theta = 2\varphi(\overline{\theta}), \tag{7.20}$$

$$-\frac{a_0^2}{2} + \rho_2(\overline{\theta}) + k_2 \int_{S_1} \rho_1(\theta)(f(\theta,\overline{\theta}))^4 d\theta + k_3 \int_{S_1} \rho_1(\theta)(f(\theta,\overline{\theta}))^2 d\theta = 0, \tag{7.21}$$

donde θ y $\overline{\theta}$ están asociados con x e y, respectivamente y

$$f(\theta,\overline{\theta}) = \frac{1}{\sqrt{R_1^2 + R_2^2 - 2R_1R_2\cos(\theta-\overline{\theta})}},$$

la cual es la expresión en coordenadas polares para $\frac{1}{|x-y|}$.

7.5.1 Ejemplo numérico

En esta sección ilustramos, mediante un ejemplo en el caso de una región anular circular, la técnica anterior para encontrar la solución clásica (que en este ejemplo coincide con la débil) del problema auxiliar dado por (7.2) con el ejemplo siguiente. Mediante un programa desarrollado en MATLAB calculamos la solución aproximada del sistema de ecuaciones integrales (7.20)-(7.21) para una condición de contorno $\varphi(\theta) = R_1\cos\theta$ y para $R_1 = 1$, $R_2 = 5$. Para hallar la solución aproximada truncamos las series (7.18) y (7.19) y utilizamos una partición regular de $[0, 2\pi]$ para los puntos de colocación. Si truncamos en $s+1$ términos cada serie, entonces necesitamos $n = 2s+2$ puntos de colocación, para obtener un sistema cuadrado de ecuaciones lineales.

La solución exacta para este ejemplo puede calcularse con armónicos circulares y tiene la expresión siguiente:

$$u(r,\theta) = \left[\frac{R_1^2 r}{R_1^2+R_2^2} + \frac{R_1^2R_2^2}{\left(R_1^2+R_2^2\right)r}\right]\cos\theta. \tag{7.22}$$

En la figura (7.1) se muestra la comparación entre la solución exacta y la solución aproximada para el ejemplo mencionado. El error relativo entre la solución exacta y la solución aproximada es en este caso 4.1×10^{-2}. Debido a que el problema auxiliar es bien planteado, no hay necesidad de emplear alguna técnica de regularización.

Si buscamos la solución del problema inverso en la forma (7.6), podemos hacer un análisis análogo al realizado en la demostración del teorema (2) y obtener la ecuación operacional

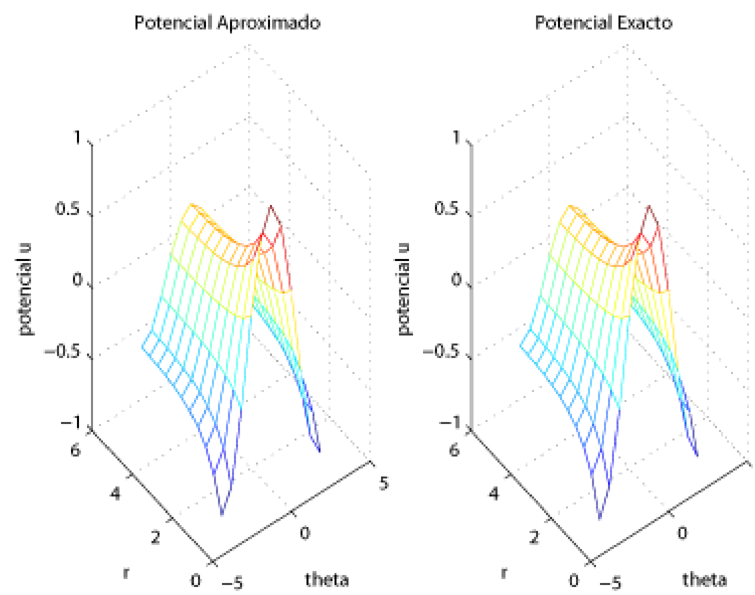

Figure 7.1: Comparación de las soluciones aproximada y exacta

$$(\overline{K} + I_0)(\rho) = \overline{v}, \tag{7.23}$$

donde

$$\overline{K} = \begin{bmatrix} \overline{K}_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix},$$

$I_0 = [0 \quad I_2]$ con I_2 denotando el operador identidad que actúa en el espacio $C(S_2)$, $\overline{K}_{11}$ es el operador dado por (7.9) sin el valor principal y los operadores $K_{12},\ K_{21},\ K_{22}$ están dados por (7.10), (7.11) y (7.12). Sin embargo, la ecuación (7.23) no es una ecuación de Fredholm de segunda especie, pues I_0 no es el operador identidad en el espacio $C(S_1) \times C(S_2)$. En este caso, el operador I_0 se obtiene al buscar u en la forma (7.6), ya que el potencial de capa doble no presenta salto en S_2, debido a que se define sobre S_1.

Como se ha mencionado, el problema inverso consiste en encontrar φ cuando se conoce v, es decir, dada v tenemos que encontrar las densidades ρ_1 y ρ_2, construir la solución u con estas densidades y después restringir u a S_1 para encontrar φ. El mal planteamiento de este problema se presenta al buscar las densidades, precisamente por el hecho de que la ecuación operacional asociada al problema inverso es la ecuación (7.23) y esto se reflejará en el sistema de ecuaciones asociado.

Los programas deben modificarse para este problema para que incluyan algún método de regularización como el de Tijonov o el de Lavrentiev. Este es un trabajo que se realizará posteriormente.

7.6 Conclusiones

1. En este trabajo se estudió un problema de contorno auxiliar asociado al problema de Cauchy para la ecuación de Laplace en una región anular sobre el plano el cual permite establecer un planteamiento operacional. La existencia y unicidad de las soluciones clásica y débil para el problema auxiliar se analizaron.

2. Se demostró que el problema auxiliar es equivalente a un sistema de ecuaciones integrales y que las soluciones clásica y débil pueden buscarse como una suma de potenciales de superficie, lo que permitió proponer un algoritmo para hallar la solución del problema.

3. Para el caso de una región anular circular se desarrolló este algoritmo para encontrar una solución aproximada del problema auxiliar, el cual puede extenderse a regiones más generales. Se validaron los resultados teóricos por medio de un ejemplo sintético.

4. Para el problema de Cauchy puede proponerse un algoritmo para hallar una solución aproximada siguiendo la técnica desarrollada para el problema auxiliar. Esto no se desarrolló en este trabajo.

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8

SVEP, Browder and Weyl theorems

B. P. Duggal

Abstract: Browder/Weyl theorems, in their classical and more recently in their generalized form, have been considered by a large number of authors. This paper brings forth the important role played by SVEP (the single–valued extension property) and the polaroid property (isolated points of the spectrum of an operator are poles of the resolvent of the operator) in determining when an operator satisfies Weyl's theorem.

Keywords: Banach space, single valued extension property, Browder and Weyl spectra, Browder and Weyl theorems.

AMS subject: Primary 47B47, 47A10, 47A11.

8.1. Introduction

An operator $T \in B(\mathcal{X})$, the algebra of bounded linear operators on an infinite dimensional complex Banach space $\mathcal{X}$ into itself, has *the single-valued extension property* at a point $\lambda_0 \in \mathbf{C}$, SVEP at λ_0, if for every open disc $\mathcal{D}$ centered at λ_0 the only analytic function $f : \mathcal{D} \longrightarrow \mathcal{X}$ satisfying $(T - \lambda I)f(\lambda) = 0$ is the function $f \equiv 0$. Evidently, every T has SVEP at points in the resolvent $\rho(T) = \mathbf{C} \setminus \sigma(T)$ or the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$ of T. Here $\mathbf{C}$ denotes the set of complex numbers. It is easily verified that SVEP is inherited by restrictions. We say that T has SVEP if it has SVEP at every $\lambda \in \sigma(T)$. Ever since its introduction by Dunford [18], [19] during the early fifties of the last century, SVEP has played an important role in *local spectral theory* and *Fredholm theory*; the interested reader is referred to [30] and [2] for further information on SVEP and the important position it holds in the study of local spectral theory and Fredholm theory.

An operator $T \in B(\mathcal{X})$ is said to be Fredholm, $T \in \Phi(\mathcal{X})$, if the range $T\mathcal{X}$ of T is closed and both its *deficiency indices* $\alpha(T) = \dim(T^{-1}(0))$ and $\beta(T) = \dim(\mathcal{X}/T\mathcal{X})$ are finite, and then the *index* of T, $\mathrm{ind}(T)$, is defined to be $\mathrm{ind}(T) = \alpha(T) - \beta(T)$. The ascent of T, $\mathrm{asc}(T)$, is the least non-negative integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$ and the descent of T, $\mathrm{dsc}(T)$, is the least non-negative integer n such that $T^n(\mathcal{X}) = T^{n+1}(\mathcal{X})$; if no such number exists, then the ascent, respectively the descent, of T is defined to be infinite. *We say that T is of finite ascent (resp., finite descent) if $asc(T - \lambda I) < \infty$ (resp., $dsc(T - \lambda I) < \infty$) for all complex numbers λ.* We shall, henceforth, shorten $(T - \lambda I)$ to $(T - \lambda)$. The operator T is *Weyl* if it is Fredholm of index zero , and T is said to be *Browder* if it is Fredholm "of finite ascent and descent". The Browder spectrum $\sigma_b(T)$ and the Weyl spectrum $\sigma_w(T)$ of T are the sets $\sigma_b(T) = \{\lambda \in \mathbf{C} : T - \lambda \quad \text{is not Browder}\}$ and $\sigma_w(T) = \{\lambda \in \mathbf{C} : T - \lambda \ \text{ is not Weyl}\}$. The (Fredholm) essential spectrum $\sigma_e(T)$ of $T \in B(\mathcal{X})$ is the set $\sigma_e(T) = \{\lambda \in \mathbf{C} : T - \lambda \notin \Phi(\mathcal{X})\}$. If we let $\mathrm{acc}\sigma(T)$ denote the set of accumulation points of $\sigma(T)$, then

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) \subseteq \sigma_e(T) \cup \mathrm{acc}\sigma(T).$$

Let $\Phi_{SF_+}(\mathcal{X}) = \{T \in B(\mathcal{X}) : \alpha(T) < \infty \ \text{ and } \ T(\mathcal{X}) \ \text{ is closed}\}$ and

$$\Phi_{SF_-}(\mathcal{X}) = \{T \in B(\mathcal{X}) : \beta(T) < \infty\}$$

denote, respectively, the semi-groups of *upper semi–Fredholm* and *lower semi–Fredholm* operators in $B(\mathcal{X})$, and let $\Phi_{SF}(\mathcal{X}) = \Phi_{SF_+}(\mathcal{X}) \cup \Phi_{SF_-}(\mathcal{X})$.

Let $\Phi_{SF_+^-}(\mathcal{X}) = \{T \in \Phi_{SF_+}(\mathcal{X}) : \operatorname{ind}(T) \leq 0\}$, and let $\Phi_{SF_-^+}(\mathcal{X}) = \{T \in \Phi_{SF_-}(\mathcal{X}) : \operatorname{ind}(T) \geq 0\}$. The *essential approximate point spectrum* (resp., *the essential Browder approximate point spectrum*) of T is the set $\sigma_{aw}(T) = \cap\{\sigma_a(T+K) : K \in K(\mathcal{X})\} = \{\lambda \in \sigma_a(T) : (T-\lambda) \notin \Phi_+^-(\mathcal{X}\})$ (resp., $\sigma_{ab}(T) = \cap\{\sigma_a(T+K) : TK = KT, K \in K(\mathcal{X})\} = \{\lambda \in \sigma_a(T) : T-\lambda \notin \Phi_+^-(\mathcal{X}), \operatorname{asc}(T-\lambda) = \infty\})$ [2]. Here $K(\mathcal{X}) \subset B(\mathcal{X})$ denotes the ideal of compact operators, and $\sigma_a(T)$ denotes *the approximate point spectrum* of T. Evidently, $\sigma_{aw}(T) \subseteq \sigma_{ab}(T)$.

Let $\mathcal{R}(T) = \{\lambda \in \operatorname{iso}\sigma(T) : \operatorname{asc}(T-\lambda) = \operatorname{dsc}(T-\lambda) < \infty\}$ denote the set of *poles of the resolvent* of T, $\mathcal{R}_0(T) = \{\lambda \in \mathcal{R}(T) : T-\lambda \in \Phi(\mathcal{X})\}$ denote the set of *Riesz points* of T, $\Pi(T) = \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(T-\lambda)\}$ denote the set of isolated eigenvalues of T, and let $\Pi_0(T) = \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(T-\lambda) < \infty\}$ denote the set of isolated eigenvalues of T of finite geometric multiplicity. In common with current terminology, we say that an operator $T \in B(\mathcal{X})$ satisfies *Browder's theorem* (resp., *a-Browder's theorem*), or Bt (resp., $a-Bt$), if $\operatorname{acc}\sigma(T) \subseteq \sigma_w(T)$ (resp., $\operatorname{acc}\sigma_a(T) \subseteq \sigma_{aw}(T)$); T satisfies *Weyl's theorem* (resp., *a-Weyl's theorem*), or Wt (resp., $a-Wt$), if $\sigma(T) \setminus \sigma_w(T) = \Pi_0(T)$ (resp., $\sigma_a(T) \setminus \sigma_{aw}(T) = \Pi_0^a(T)$).

The purpose of this paper is to show the important role that SVEP (at points in the complement of some of the distinguished parts of the spectrum) plays in determining when an operator $T \in B(\mathcal{X})$ satisfies Bt or $a-Bt$, and the important role that SVEP coupled with an additional property (the so called polaroid property at isolated eigenvalues of $\sigma(T)$ or $\sigma_a(T)$) plays in determining when an operator $T \in B(\mathcal{X})$ satisfies Wt or $a-Wt$. The (more classical) notion of Browder and Weyl theorem has been generalized by Berkani and a host of other authors (see [4,6,7,8,9,10,14,19,21]). Here too, SVEP (also, the polaroid property) plays an important role. Bt and $a-Bt$ have a permanence property: $T \in B(\mathcal{X})$ satisfies Bt if and only if it satisfies generalized Browder's theorem, and T satisfies $a-Bt$ if and only if it satisfies generalized a-Browder's theorem. Wt and $a-Wt$ have no such permanence property: generalized Weyl's theorem (resp., generalized a-Weyl's theorem) implies WT (resp., $a-Wt$), but the converse is false. We prove however that there are classes of operators in $B(\mathcal{X})$ for which Wt shares this permanence property with Bt. We also consider Browder and Weyl theorems for perturbations of operators. We shall, in particular, exhibit a large class of operators which satisfy Wt if and only if their perturbation by a commuting algebraic operator satisfies Wt.

The plan of this paper is as follows. In addition to the notation and terminology already introduced, we introduce terminology, and prove some lemmas, pertinent to generalized Browder and Weyl theorems in the following section; any other notation or terminology will be introduced on an as and when required basis. Section 3 considers Browder's theorem, Section 4 considers Weyl's theorem, and Section 5, the final section, is devoted to examples. Except for a few, most of the results of this paper are well known, and spread over a large number of papers; however, the presentation here is new. We list some of these papers, indeed a small fraction of the papers available in the extant literature, at the end of the paper. No attempt has been made to attribute individual results to their source: my apologies for this.

8.2. Notation and terminology

For an operator $T \in B(\mathcal{X})$ such that $T^n\mathcal{X}$ is closed for some $n \in \mathbf{N}$, let $T_{[n]}$ ($T_{[0]} = T$) denote the induced mapping

$$T_{[n]} = T|_{T^n\mathcal{X}} \in B(T^n\mathcal{X}).$$

Following Berkani [6] we say that the operator T is

$$\begin{aligned} &\text{semi B–Fredholm}, T \in \Phi_{SBF}(\mathcal{X}), \textit{if } T_{[n]} \in \Phi_{SF}(\mathcal{X}); \\ &\text{B–Fredholm}, T \in \Phi_{BF}(\mathcal{X}), \textit{if } T_{[n]} \in \Phi(\mathcal{X}). \end{aligned}$$

Observe that $T_{[n]} \in \Phi(\mathcal{X})$ for some $n \in \mathbf{N}$ implies $T_{[m]} \in \Phi(\mathcal{X})$ for all $m \geq n$. Hence it makes sense to define (and we do so) the index of T by $\operatorname{ind}(T) = \operatorname{ind}(T_{[n]})$. An operator T is said to be of *uniform descent for some* $d \in \mathbf{N}$ if $T\mathcal{X} + T^{-m}(0) = T\mathcal{X} + T^{-d}(0)$ for all $m \geq d$; if additionally $T\mathcal{X} + T^{-d}(0)$ is closed, then T is said to be of *topological uniform descent for all* $m \geq d$. It is clear that if T is of toplogical uniform descent for some $d \in \mathbf{N}$, then T^* is of toplogical uniform descent for d. If either of the deficiency indices $\alpha(T)$ and $\beta(T)$, or the chain lengths $\mathrm{asc}(T)$ and $\mathrm{dsc}(T)$, is finite, then T has uniform descent. Let

$$\begin{aligned}
\Phi_{SBF_+^-}(T) &= \{\lambda \in \sigma(T) : (T-\lambda)_{[n]} \\
&\qquad\qquad \text{is upper semi–Fredholm and } \mathrm{ind}(T-\lambda) \leq 0\}, \\
\Phi_{SBF_-^+}(T) &= \{\lambda \in \sigma(T) : (T-\lambda)_{[n]} \\
&\qquad\qquad \text{is lower semi–Fredholm and } \mathrm{ind}(T-\lambda) \geq 0\}, \\
\sigma_{BW}(T) &= \{\lambda \in \sigma(T) : T-\lambda \notin \Phi_{BF}(\mathcal{X}) \text{ or } \mathrm{ind}(T-\lambda) \neq 0\}, \\
\sigma_{SBW_+^-}(T) &= \{\lambda \in \sigma(T) : T-\lambda \notin \Phi_{SBF_+^-}(\mathcal{X})\}, \\
\sigma_{SBW_-^+}(T) &= \{\lambda \in \sigma(T) : T-\lambda \notin \Phi_{SBF_-^+}(\mathcal{X})\} \\
&= \{\lambda \in \sigma(T) : T^* - \lambda I^* \notin \Phi_{SBF_+^-}(\mathcal{X}^*)\}.
\end{aligned}$$

Then $\sigma_{BW}(T) = \sigma_{SBW_+^-}(T) \cup \sigma_{SBW_-^+}(T) = \sigma_{SBW_-^+}(T^*) \cup \sigma_{SBW_+^-}(T^*)$. Let $\mathcal{P}^a(T) = \{\lambda \in \sigma_a(T) : \mathrm{asc}(T-\lambda) = d < \infty, T^{d+1}\mathcal{X} \text{ is closed}\}$ denote the set of *left poles of* T *of some order* $d \in \mathbf{N}$, and let $\mathcal{P}^s(T) = \{\lambda \in \sigma_s(T) : \mathrm{dsc}(T-\lambda) = d < \infty, T^d\mathcal{X} \text{ is closed}\}$ denote the set of *right poles of* T *of some order* $d \in \mathbf{N}$. Here $\sigma_s(T)$ denote the *surjectivity spectrum of* T.

The following lemmas relating toplogical uniform descent, SVEP, ascent, $\mathcal{P}^a(.)$, and $\sigma_{SBW_+^-}(.)$ are well known (see [7],[4],[16]). We include a proof for completeness.

Lemma 8.2.1 *If* T *is an operator of topological uniform descent, then* T *has SVEP at* $0 \Longleftrightarrow$ *asc*$(T) < \infty$.

Proof. Since $\mathrm{asc}(T) < \infty \Longrightarrow T$ has SVEP at 0, we prove the converse. Suppose that T has uniform topological descent for $n \geq d$ $(\in \mathbf{N})$. Letting $Y = \cap_{n \geq d} T^n\mathcal{X}$, it is seen that Y is a T invariant closed subspace of $T^d\mathcal{X}$ such that $T|_Y$ is onto [22, Theorem 3.4]. Since T has SVEP at 0, $T|_Y$ has SVEP at 0, which (since a surjective operator has SVEP at a point if and only if it is injective at the point [2, Corollary 2.24]) implies $T^{-n}(0) \cap T^d\mathcal{X} = \{0\}$ for all $n \geq d$. Hence $\mathrm{asc}(T) \leq d$ [2. p. P 111]. □

By duality, Lemma 8.2.1 implies that *if* T *is an operator of topological uniform descent, then* T^* *has SVEP at* $0 \Longleftrightarrow$ *dsc*$(T) < \infty$.

Lemma 8.2.2 *If $\lambda \in \mathcal{P}^a(T)$, then $T-\lambda$ is of topological uniform descent, $\lambda \in iso\sigma_a(T)$ and $\lambda \notin \sigma_{SBW_+^-}(T)$.*

Proof. We may assume, without loss of generality, that $\lambda = 0$. If $\mathrm{asc}(T) = d < \infty$ and $T^{d+1}\mathcal{X}$ is closed, then, for all $n \geq d$, $T\mathcal{X} + (T)^{-d}(0) = T\mathcal{X} + (T)^{-n}(0)$ is closed [30, Proposition 4.10.4]; hence T is of topological uniform descent. Assume now that $0 \notin \mathrm{iso}\sigma_a(T)$. Then there exists a sequence $\{\lambda_n\} \subset \sigma_p(T)$ such that $\lambda_n \neq 0$ for all n and $\lim_{n\longrightarrow\infty} \lambda_n = 0$. Since T has topological uniform descent for $n \geq d$, there exists an $\epsilon > 0$ such that $(T-\mu)\mathcal{X}$ is closed for all $0 < |\mu| < \epsilon$; hence $\mu \in \sigma_a(T)$ if and only if $\mu \in \sigma_p(T)$. Furthermore, since $(T-\mu)^{-1}(0) \subseteq T^\infty\mathcal{X}(= \cap_{n=1}^\infty T^n\mathcal{X})$ for all $\mu \neq 0$, it follows that $\lambda_n \in \sigma_a(T_1)$, where $T_1 = T|_{T^\infty\mathcal{X}}$. Since $\sigma(T_1)$ is closed, $0 \in \sigma(T_1)$. The hypothesis $\mathrm{asc}(T) < \infty \Longrightarrow T$ has SVEP at $0 \Longrightarrow T_1$ has SVEP at 0. Already, T_1 is onto [22, Theorem 3.4]; hence the fact that T_1 has SVEP at 0 implies that T_1 is injective [2, Corollary 2.24], which in turn implies that $0 \notin \sigma(T_1)$, a contradiction. Thus $0 \in \mathrm{iso}\sigma_a(T)$. To complete the proof, we observe that $\mathrm{asc}(T) = d < \infty \Longrightarrow \mathrm{ind}(T) \leq 0$ and $T^n\mathcal{X} \cap (T)^{-1}(0) = \{0\}$ [30, Lemma 4.10.1]. Hence $\alpha(T|_{T^d\mathcal{X}}) < \infty$ and $\mathrm{ind}(T) \leq 0$. Since $T^n\mathcal{X}$ is closed for all $n \geq d$, $0 \notin \sigma_{SBW_+^-}(T)$. $\square$

Using Banach space duality, Lemma 8.2.2 implies the following.

Corollary 8.2.3 *If $\lambda \in \mathcal{P}^s(T)$, then $\lambda \in iso\sigma_s(T)$ and $\lambda \notin \sigma_{SBW_-^+}(T)$.*

The following lemma gives a sufficient condition for $\sigma_{SBW_+^-}(T) = \sigma_{BW}(T)$.

Lemma 8.2.4 *If T has SVEP at points $\lambda \notin \sigma_{SBW_-^+}(T)$ (resp., T^* has SVEP at points $\lambda \notin \sigma_{SBW_+^-}(T)$), then $\sigma_{SBW_-^+}(T) = \sigma_{BW}(T)$ (resp., $\sigma_{SBW_+^-}(T) = \sigma_{BW}(T)$).*

Proof. If $\lambda \notin \sigma_{SBW_-^+}(T)$ (or $\lambda \notin \sigma_{SBW_+^-}(T)$), then $T-\lambda$ has topological uniform descent. Thus, if T has SVEP at λ (resp., T^* has SVEP at λ), then $\mathrm{asc}(T-\lambda) < \infty$ (resp., $\mathrm{dsc}(T-\lambda) < \infty$). Hence, $\mathrm{ind}(T-\lambda) \leq 0$ (resp., $\mathrm{ind}(T-\lambda) \geq 0$). Since already $\mathrm{ind}(T-\lambda) \geq 0$ (resp., $\mathrm{ind}(T-\lambda) \leq 0$), $\mathrm{ind}(T-\lambda) = 0$ and $\lambda \notin \sigma_{BW}(T)$; hence $\sigma_{BW}(T) \subseteq \sigma_{SBW_-^+}(T)$ and, respectively, $\sigma_{BW}(T) \subseteq \sigma_{SBW_+^-}(T)$. The reverse inclusions being always true, the lemma is proved. $\square$

Let $LD(\mathcal{X})$ denote the *regularity*

$$LD(\mathcal{X}) = \{T \in B(\mathcal{X}) : \operatorname{asc}(T) = d < \infty, T^{d+1}(\mathcal{X}) \text{ is closed}\},$$

and let

$$\sigma_{LD}(T) = \{\lambda \in \mathbf{C} : T - \lambda \notin LD(\mathcal{X})\}$$

denote the spectrum induced by the regularity $LD(\mathcal{X})$. (The interested reader is invited to consult [31] and [32] for information on regularities and the spectra induced by regularities.) Lemma 8.2.1 implies that *if T has SVEP at points $\lambda \notin \sigma_{SBW_+^-}(T)$, then $\sigma_{LD}(T) = \sigma_{SBW_+^-}(T)$.* (Indeed, $\sigma_{LD}(T) = \sigma_{SBW_+^-}(T)$ if and only if T has SVEP at points $\lambda \notin \sigma_{SBW_+^-}(T)$.) Since every regularity satisfies the spectral mapping theorem for functions $f \in H_c(\sigma(T))$, if T has SVEP at points $\lambda \notin \sigma_{SBW_+^-}(T)$, then $f(\sigma_{SBW_+^-}(T)) = \sigma_{SBW_+^-}(f(T))$ for every $f \in H_c(\sigma(T))$. Here, and in the sequel, $H(\sigma(T))$ denotes the class of (non–trivial) functions f which are analytic on a neighbourhood of $\sigma(T)$ and $H_c(\sigma(T))$ denotes the class of $f \in H(\sigma(T))$ which are non–constant on every connected component of $\sigma(T)$.

Corollary 8.2.5 *If T has SVEP at points $\lambda \notin \sigma_{SBW_+^-}(T)$, then*

$$f(\sigma_{SBW_+^-}(T)) = \sigma_{SBW_+^-}(f(T))$$

for every $f \in H_c(\sigma(T))$.

Let $RD(\mathcal{X})$ denote the regularity $RD(\mathcal{X}) = \{T \in B(\mathcal{X}) : \operatorname{dsc}(T) = d < \infty, T^d\mathcal{X} \text{ is closed}\}$. Then $D(\mathcal{X}) = LD(\mathcal{X}) \cap RD(\mathcal{X})$ is a regularity. Let $\sigma_D(T) = \{\lambda \in \sigma(T) : \lambda \notin D(\mathcal{X})\}$. $\sigma_D(T)$ ($\supseteq \sigma_{BW}(T)$) is then the *Drazin spectrum* of T. Evidently, $\sigma_D(T)$ satisfies $f(\sigma_D(T)) = \sigma_D(f(T))$ for every $f \in H_c(\sigma(T))$.

Lemma 8.2.6 *If $T \in B(\mathcal{X})$ has SVEP at points $\lambda \notin \sigma_{BW}(T)$, then*

$$f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$$

for every $f \in H_c(\sigma(T))$.

Proof. The proof of the lemma follows from the observation (proved below) that $\sigma_{BW}(T) = \sigma_D(T)$ if and only if T has SVEP at points $\lambda \notin \sigma_{BW}(T)$. If $\lambda \notin \sigma_{BW}(T)$, then $(T-\lambda)_{[n]} \in \Phi(\mathcal{X})$ and $\text{ind}((T-\lambda)_{[n]}) = 0$ for some $n \in \mathbf{N}$. This, if T has SVEP at points $\lambda \in \Phi_{BF}(T)$, implies that $\text{asc}((T-\lambda)_{[n]}) = \text{dsc}((T-\lambda)_{[n]}) < \infty$ (by Lemma 8.2.4 and [2, Theorem 3.4]). Thus $\text{asc}(T - \lambda) = \text{dsc}(T-\lambda) < \infty \Longrightarrow \lambda \notin \sigma_D(T)$. Consequently $\sigma_D(T) \subseteq \sigma_{BW}(T)$, which implies that $\sigma_{BW}(T) = \sigma_D(T)$. Evidently, if $\sigma_{BW}(T) = \sigma_D(T)$, then T has SVEP at points $\lambda \notin \sigma_{BW}(T)$. $\square$

Assuming a bit more, Lemma 8.2.6 extends to all $f \in H(\sigma(T))$ as follows.

Lemma 8.2.7 *If T has SVEP at points $\lambda \in \Phi_{BF}(T)$, then $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$ for every $f \in H(\sigma(T))$.*

Proof. The hypotheses imply that $\sigma_{BW}(T) = \sigma_D(T)$. Hence, by the spectral mapping theorem for $\sigma_D(T)$ [9], it follows that $f(\sigma_{BW}(T)) = f(\sigma_D(T)) = \sigma_D(f(T))$ $(\supseteq \sigma_{BW}(f(T)))$ for every $f \in H(\sigma(T))$. Suppose now that $\mu \notin \sigma_{BW}(f(T))$. Then $f(T) - \mu \in \Phi_{BF}(\mathcal{X})$ and $\text{ind}(f(T) - \mu) = 0$. Since $f(\Phi_{BF}(T)) = \Phi_{BF}(f(T))$ for every $f \in H(\sigma(T))$ [6, Theorem 3.4], there exists a $\lambda \in \Phi_{BF}(T)$ such that $\mu = f(\lambda)$. Since T has SVEP at λ, $f(T)$ has SVEP at μ. Hence $\mu \notin \sigma_D(f(T)) = (f(\sigma_{BW}(T)))$. Thus $f(\sigma_{BW}(T)) \subseteq \sigma_{BW}(f(T))$. $\square$

Lemma 8.2.8 *If T has SVEP at points $\lambda \in \Phi_{SBF_-}(T)$ (resp., T^* has SVEP at points $\lambda \in \Phi_{SBF_+}(T)$), then $f(\sigma_{SBW_-^+}(T)) = \sigma_{SBW_-^+}(f(T))$ (resp., $f(\sigma_{SBW_+^-}(T)) = \sigma_{SBW_+^-}(f(T))$ for every $f \in H_c(\sigma(T))$.*

Proof. Observe that if $\lambda \in \Phi_{SBF_-}(T)$ (resp., $\lambda \in \Phi_{SBF_+}(T)$), then $T - \lambda \in LD(\mathcal{X})$ (resp., $T - \lambda \in RD(\mathcal{X})$) if and only if T has SVEP at λ (resp., T^* has SVEP at λ). The proof thus follows from the fact that the regularities $LD(\mathcal{X})$ and $RD(\mathcal{X})$ satisfy the spectral mapping theorem for every $f \in H_c(\sigma(T))$. $\square$

8.3. Browder's theorem

We say that T satisfies:

$$\text{Browder's theorem, or } Bt, \text{if } \operatorname{acc}\sigma(T) \subseteq \sigma_w(T);$$
$$a-\text{Browder's theorem, or } a-Bt, \text{if } \operatorname{acc}\sigma_a \subseteq \sigma_{aw}(T).$$

For $T \in B(\mathcal{X})$, let

$$\mathcal{R}(T) = \{\lambda \in \operatorname{iso}\sigma(T) : \operatorname{asc}(T-\lambda) = \operatorname{dsc}(T-\lambda) < \infty\};$$
$$\mathcal{R}^a(T) = \{\lambda \in \operatorname{iso}\sigma_a(T) : \operatorname{asc}(T-\lambda) = d < \infty, (T-\lambda)\mathcal{X} \text{ is closed}\};$$
$$\mathcal{R}_0(T) = \{\lambda \in \mathcal{R}(T) : \alpha(T-\lambda) < \infty\};$$
$$\mathcal{R}_0^a(T) = \{\lambda \in \mathcal{R}^a(T) : \alpha(T-\lambda) < \infty\}.$$

Let $\gamma(T)$ denote the *reduced minimum modulus* of T, and let $\triangle(T) = \{\lambda \in \mathbf{C} : \lambda \notin \sigma_w(T), 0 < \alpha(T-\lambda)\}$ (denote the set of *generalized Riesz points* of T).

Before going on to prove a number of equivalent conditions characterizing operators satisfying Bt (or $a-Bt$ or their generalized versions), we recall the following well known facts (see [2]). The *quasinilpotent part* $H_0(T-\lambda)$ and the *analytic core* $K(T-\lambda)$ of $(T-\lambda)$ are defined by

$$H_0(T-\lambda) = \{x \in \mathcal{X} : \lim_{n\longrightarrow\infty} ||(T-\lambda)^n x||^{\frac{1}{n}} = 0\}$$

and

$$\begin{aligned} K(T-\lambda) = \{x \in \mathcal{X} : &\text{there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0 \\ &\text{for which } x = x_0, (T-\lambda)(x_{n+1}) = x_n \text{ and } \|x_n\| \le \delta^n \|x\| \\ &\text{for all } n = 1, 2, ...\}. \end{aligned}$$

$H_0(T-\lambda)$ and $K(T-\lambda)$ are (generally) non-closed hyperinvariant subspaces of $(T-\lambda)$ such that $(T-\lambda)^{-q}(0) \subseteq H_0(T-\lambda)$ for all $q = 0, 1, 2, ...$ and $(T-\lambda)K(T-\lambda) = K(T-\lambda)$; also, if $\lambda \in \operatorname{iso}\sigma(T)$, then $\mathcal{X} = H_0(T-\lambda) \oplus K(T-\lambda)$. If $T-\lambda \in \Phi_{SF}(\mathcal{X})$ and T (resp., T^*) has SVEP at λ, then $\operatorname{asc}(T-\lambda) < \infty$ and $\lambda \in \operatorname{iso}\sigma_a(T)$ (resp., $\operatorname{dsc}(T-\lambda) < \infty$ and $\lambda \in \operatorname{iso}\sigma_s(T) = \operatorname{iso}\sigma_a(T^*)$); if $T-\lambda \in \Phi(\mathcal{X})$, $\operatorname{ind}(T-\lambda) = 0$ and either T or T^* has SVEP at λ, then

$\lambda \in \mathcal{R}(T)$; $\text{asc}(T-\lambda) < \infty \Longrightarrow \alpha(T-\lambda) \leq \beta(T-\lambda)$ and $\text{dsc}(T-\lambda) < \infty \Longrightarrow \beta(T-\lambda) \leq \alpha(T-\lambda)$. An operator $T \in B(\mathcal{X})$ is said to be *Kato type at a point* λ if there exists a pair of T invariant (closed) subspaces $\mathcal{X}_1$ and $\mathcal{X}_2$ of $\mathcal{X}$ such that $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$, $T_1 = (T-\lambda)|_{\mathcal{X}_1}$ is nilpotent and $T_2 = (T-\lambda)|_{\mathcal{X}_2}$ is semi–regular (i.e., it satisfies the properties that $T_2^n\mathcal{X}_2$ is closed and $T_2^{-1}(0) \subseteq T_2^n\mathcal{X}_2$ for all $n \in \mathbf{N}$) [2, p. 24]. A semi–regular T has SVEP at 0 precisely when it is injective (equivalently, bounded below) [2, Theorem 2.49]. $T \in B(\mathcal{X})$ is said to be *essentially semi–regular* at λ if there exists a pair of T invariant (closed) subspaces $\mathcal{X}_1$ and $\mathcal{X}_2$ of $\mathcal{X}$ such that $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$, $\dim(\mathcal{X}_1) < \infty$, $(T-\lambda)|_{\mathcal{X}_1}$ is quasinilpotent (hence, nilpotent) and $(T-\lambda)|_{\mathcal{X}_2}$ is semi–regular. Operators $T \in \Phi_{\pm}(\mathcal{X})$ are Kato type and essentially semi–regular. For an essentially semi–regular operator $T-\lambda$, the property that $\dim H_0(T-\lambda) < \infty$ is equivalent to the statement that T has SVEP at λ [2, Theorem 3.18]. We shall use these results in the sequel without any further reference.

The following theorem characterizes operators $T \in B(\mathcal{X})$ satisfying Bt. Most of the conditions listed in the theorem are well known (and are spread over a number of papers, see for example [2],[4], [5], [7], [8], [10], [11], [12], [14], [15], [16], [17], [23] and [34).

Theorem 8.3.1 *The following conditions are equivalent.*
(i) T *satisfies* Bt.
(ii) $\sigma(T) = \sigma_w(T) \cup iso\sigma(T)$.
(iii) $\sigma_b(T) = \sigma_w(T)$.
(iv) T *has SVEP at points* $\lambda \notin \sigma_w(T)$.
(v) $asc(T-\lambda) < \infty$ *at points* $\lambda \notin \sigma_w(T)$.
(vi) $\dim H_0(T-\lambda) < \infty$ *at points* $\lambda \notin \sigma_w(T)$.
(vii) $H_0(T-\lambda)$ *is closed at points* $\lambda \notin \sigma_w(T)$.
(viii) $codim K(T-\lambda) < \infty$ *at points* $\lambda \notin \sigma_w(T)$.
(ix) Every $\lambda \in \triangle(T)$ *is an isolated point of* $\sigma(T)$.
(x) The mapping $\lambda \longrightarrow \gamma(T-\lambda)$ *is not continuous at every* $\lambda \in \triangle(T)$.
(xi) $\sigma(T) \setminus \sigma_w(T) = \mathcal{R}_0(T)$.

Proof. (i)$\Longrightarrow$(ii). Evidently,

$$\sigma(T) = \text{acc}\sigma(T) \cup \text{iso}\sigma(T) \subseteq \sigma_w(T) \cup \text{iso}\sigma(T) \subseteq \sigma(T).$$

(ii)$\Longrightarrow$(iii). If $\lambda \in \sigma(T) \setminus \sigma_w(T)$, then $T - \lambda \in \Phi(\mathcal{X})$, $ind(T-\lambda) = 0$ and T has SVEP at $\lambda \Longrightarrow T - \lambda \in \Phi(\mathcal{X})$ and $\text{asc}(T-\lambda) = \text{dsc}(T-\lambda) < \infty \Longrightarrow \lambda \notin \sigma_b(T) \Longrightarrow \sigma_b(T) \subseteq \sigma_w(T)$. Since $\sigma_w(T) \subseteq \sigma_b(T)$ for every T, $\sigma_w(T) = \sigma_b(T)$.

(iii)$\Longrightarrow$ (iv)$\Longrightarrow$ (v). Evident, since $\sigma_b(T) = \sigma_w(T) \Longleftrightarrow \text{asc}(T-\lambda) < \infty$ at points $\lambda \notin \sigma_w(T) \Longleftrightarrow T$ has SVEP at points $\lambda \notin \sigma_w(T)$.

(v) $\Longrightarrow$ (vi) $\Longrightarrow$ (vii)$\Longrightarrow$ (viii). That (v) implies (vi) implies (vii) follows from the fact that $H_0(T-\lambda) = (T-\lambda)^{-p}(0)$ for some $p \in \mathbf{N}$ at points $\lambda \notin \sigma_w(T)$ such that T has SVEP at λ. Since (vii) implies that T has SVEP at λ, $K(T-\lambda) = (T-\lambda)^p\mathcal{X}$ for some $p \in \mathbf{N}$ is finite co–dimensional. Hence (vii) implies (viii).

(viii) $\Longrightarrow$ (ix). If (viii) holds and $\lambda \in \triangle(T)$, then $\lambda \notin \sigma_w(T)$ and $\text{codim}K(T-\lambda) < \infty$ imply that T^* has SVEP at λ. Hence, $\lambda \in \mathcal{R}_0(T)$; in particular, $\lambda \in \text{iso}\sigma(T)$.

(ix) $\Longrightarrow$ (x). If (ix) is satisfied , then $(T-\lambda)\mathcal{X}$ is closed for every $\lambda \in \triangle(T)$. Since (x) holds if and only if $(T-\lambda)\mathcal{X}$ is closed at every $\lambda \in \triangle(T)$, (ix) $\Longrightarrow$ (x).

(x) $\Longrightarrow$ (xi)$\Longrightarrow$ (i). If (x) holds, then $\mu \notin \sigma_w(T)$ for every $\mu \in \triangle(T)$. By the punctured neighbourhood theorem, there exists an $\epsilon > 0$ and a neighbourhood $\mathcal{O}_\epsilon$ of μ such that $T-\lambda \in \Phi(\mathcal{X})$, $\text{ind}(T-\lambda) = \text{ind}(T-\mu) = 0$ and $\alpha(T-\lambda) < \alpha(T-\mu)$ remains constant as λ varies over $\mathcal{O}_\epsilon$. We claim that $\alpha(T-\lambda) = 0$: for if not, then there is a $\lambda_1 \in \mathcal{O}_\epsilon$ different from λ and μ such that $\alpha(T-\lambda_1) < \alpha(T-\lambda)$, which contradicts the fact that $\alpha(T-\lambda)$ remains constant over $\mathcal{O}_\epsilon$. Consequently, $\alpha(T-\lambda) = \beta(T-\lambda) = 0$, which implies that $\mu \in \text{iso}\sigma(T)$, and hence that $\mu \in \mathcal{R}_0(T)$. Since $\mu \in \mathcal{R}_0(T)$ trivially implies $\mu \notin \sigma_w(T)$, (x) $\Longrightarrow$ (xi). That (xi)$\Longrightarrow$ (i) being evident, the proof is complete. $\square$

Remark 8.3.2 It is easily seen that $\sigma(T) = \sigma(T^*)$, $\sigma_w(T) = \sigma_w(T^*)$ and $\sigma_b(T) = \sigma_b(T^*)$ for an operator $T \in B(\mathcal{X})$. Since $\lambda \in \mathcal{R}_0(T) \Longleftrightarrow \text{asc}(T - \lambda) = \text{dsc}(T - \lambda < \infty$ and $\alpha(T-\lambda) = \beta(T-\lambda) < \infty$, $\lambda \in \mathcal{R}_0(T) \Longleftrightarrow \lambda \notin \sigma_b(T) \Longleftrightarrow \lambda \notin \sigma_b(T^*) \Longleftrightarrow \lambda \notin \mathcal{R}_0(T^*)$, $\mathcal{R}_0(T) = \mathcal{R}_0(T^*)$. Hence, T *satisfies Bt if and only T^* satisfies Bt*. Evidently, conditions (i) to (xi) of Theorem 8.3.1 have their T^* counterpart: for example, T satisfies $Bt \Longleftrightarrow T^*$ satisfies $Bt \Longleftrightarrow \sigma(T^*) = \sigma_w(T^*) \cup \text{iso}\sigma(T^*) \Longleftrightarrow \sigma_b(T^*) = \sigma_w(T^*) \Longleftrightarrow T^*$

has SVEP at points $\lambda \notin \sigma_w(T^*) \Longleftrightarrow \mathrm{asc}(T^* - \lambda I^*) = \mathrm{dsc}(T - \lambda) < \infty$ at points $\lambda \notin \sigma_w(T^*)$ (etc.).

The following theorem is the analog of Theorem 8.3.1 for operators satisfying $a - Bt$. Let $\triangle^a(T) = \{\lambda \in \sigma_a(T) : T - \lambda \in \Phi_+(\mathcal{X}), \mathrm{ind}(T - \lambda) < 0\}$ $(= \sigma_a(T) \setminus \sigma_{aw}(T))$.

Theorem 8.3.3 *The following conditions are equivalent.*
(i) T satisfies $a - Bt$.
(ii) $\sigma_a(T) = \sigma_{aw}(T) \cup iso\sigma_a(T)$.
(iii) $\sigma_{ab}(T) = \sigma_{aw}(T)$.
(iv) T has SVEP at points $\lambda \notin \sigma_{aw}(T)$.
(v) $asc(T - \lambda) < \infty$ at points $\lambda \notin \sigma_{aw}(T)$.
(vi) $\dim H_0(T - \lambda) < \infty$ at points $\lambda \notin \sigma_{aw}(T)$.
(vii) $H_0(T - \lambda)$ is closed at points $\lambda \notin \sigma_{aw}(T)$.
(viii) Every $\lambda \in \triangle^a(T)$ is an isolated point of $\sigma_a(T)$.
(ix) The mapping $\lambda \longrightarrow \gamma(T - \lambda)$ is not continuous at every $\lambda \in \triangle^a(T)$.
(x) $\sigma_a(T) \setminus \sigma_{aw}(T) = \mathcal{R}_0^a(T)$.

Proof. The proof of (i)$\Longrightarrow$(ii)$\Longrightarrow$ ... $\Longrightarrow$(vii) is similar to that for the corresponding conditions of Theorem 8.3.1 (and left to the reader). If (vii) holds and $\lambda \in \triangle^a(T)$, then T has SVEP at λ (which implies that $\lambda \in \mathrm{iso}\sigma_a(T)$). Hence, (vii) implies (viii).
(viii) $\Longrightarrow$ (ix) is evident from the fact that $(T - \lambda)\mathcal{X}$ is closed for every $\lambda \in \triangle^a(T)$. To prove (ix)$\Longrightarrow$ (x), let $\mu \in \triangle^a(T)$. Then $\mu \notin \sigma_{aw}(T)$. Arguing as in the proof of (ix) implies (x) in the proof of Theorem 8.3.1, it follows that $(T - \lambda)\mathcal{X}$ is closed and $\alpha(T - \lambda) = 0$ for every λ in an ϵ-neighbourhood of μ. Hence $\mu \in \mathrm{iso}\sigma_a(T)$. But then T has SVEP at μ, which (since $\mu \notin \sigma_{aw}(A)$) implies that $\mu \in \mathcal{R}_0^a(T)$. Hence $\sigma_a(T) \setminus \sigma_{aw}(T) \subseteq \mathcal{R}_0^a(T)$. Evidently, $\mathcal{R}_0^a(T) \subseteq \sigma_a(T) \setminus \sigma_{aw}(T)$. Since (x)$\Longrightarrow$(i) is evident, the proof is complete. $\square$

T satisfies $a - Bt \not\Longrightarrow T^*$ satisfies $a - Bt$: here any operator $T \in B(\mathcal{X})$ such that T has SVEP at points $\lambda \notin \sigma_{aw}(T)$ but T^* does not have SVEP at points $\lambda \notin \sigma_{aw}(T^*)$ would do for an example. Indeed, the proper counterpart for T^* of "T satisfies $a - Bt$" is that "T satisfies $s - Bt$", where we say that

T satisfies $s-Bt$ if the accumulation points of the surjectivity spectrum $\sigma_s(T)$ of T are a subset of the set $\sigma_{sw}(T) = \{\lambda \in \sigma_s(T) : T-\lambda \notin \Phi_{SF_-}(\mathcal{X})$ or $\mathrm{ind}(T-\lambda \not\geq 0\}$. Evidently, T satisfies $a-Bt$ if and only if T^* satisfies $s-Bt$. Let

$$\sigma_{sb}(T) = \{\lambda \in \sigma_s(T) : T-\lambda \notin \Phi_{SF_-}(T) \text{ or } \mathrm{dsc}(T-\lambda) = \infty\}.$$

Then $\sigma_w(T) = \sigma_{aw}(T) \cup \sigma_{sw}(T)$ (and $\sigma_b(T) = \sigma_{ab}(T) \cup \sigma_{sb}(T)$). Hence:

Corollary 8.3.4 *Both aBt and $s-Bt$ imply Bt.*

The converse of Corollary 8.3.4 is false: consider $T = S \oplus S^* \oplus Q$, where $S \in B(\mathcal{H})$, $\mathcal{H}$ a Hilbert space, is the forward unilateral shift and $Q \in B(\mathcal{H})$ is a quasinilpotent, when it is seen that T satisfies Bt but not $a-Bt$ or $s-Bt$. Observe that if T (resp., T^*) has SVEP, then T (resp., T^*) has SVEP at points not in $\sigma_w(T)$, $\sigma_{aw}(T)$ and $\sigma_{sw}(T)$. Hence:

Corollary 8.3.5 *If T or T^* has SVEP, then both T and T^* satisfy (Bt and) $a-Bt$.*

Generalized Browder theorems. We say that T satisfies:

$$\begin{aligned}&\text{generalized Browder's theorem, or } gBt, \text{if } \mathrm{acc}\sigma(T) \subseteq \sigma_{BW}(T);\\ &\text{generalized } a-\text{Browder's theorem, or } a-gBt, \text{if}\\ &\qquad\qquad \mathrm{acc}\sigma_a(T) \subseteq \sigma_{SBW_+^-}(T).\end{aligned}$$

It is immediate from $\sigma_{BW}(T) \subseteq \sigma_w(T)$ and $\sigma_{SBW_+^-}(T) \subseteq \sigma_{aw}(T)$ that

$$gBt \Longrightarrow Bt \text{ and } a-gBt \Longrightarrow a-Bt.$$

More is true [5], as we shall prove prove in a moment. But before that we prove the following lemma.

Lemma 8.3.6 *T satisfies: (i) gBt if and only if T has SVEP at points $\lambda \notin \sigma_{BW}(T)$; (ii) $a-gBt$ if and only if T has SVEP at points $\lambda \notin \sigma_{SBW_+^-}(T)$.*

Proof. (i). If $\mathrm{acc}\sigma(T) \subseteq \sigma_{BW}(T)$, then $\lambda \notin \sigma_{BW}(T) \Longrightarrow \lambda \in \mathrm{iso}\sigma(T) \Longrightarrow T$ has SVEP at λ. Conversely, assume that T has SVEP at every $\lambda \notin \sigma_{BW}(T)$. Then there exists a $d \in \mathbf{N}$ such that $(T-\lambda)_{[d]} \in \Phi(\mathcal{X})$, $\mathrm{ind}(T-\lambda)_{[d]} = 0$ and $(T-\lambda)_{[d]}$ has SVEP at 0. Hence $\mathrm{asc}(T-\lambda)_{[d]} = \mathrm{dsc}(T-\lambda)_{[d]} < \infty$, which implies that $\mathrm{asc}(T-\lambda) = \mathrm{dsc}(T-\lambda) < \infty$, and hence that $\lambda \in \mathrm{iso}\sigma(T)$.
(ii). If $\mathrm{acc}\sigma(T) \subseteq \sigma_{SBW_+^-}(T)$, then $\lambda \notin \sigma_{SBW_+^-}(T) \Longrightarrow \lambda \in \mathrm{iso}\sigma_a(T) \Longrightarrow$ T has SVEP at λ. Conversely, assume that T has SVEP at every $\lambda \notin \sigma_{SBW_+^-}(T)$. Then there exists a $d \in \mathbf{N}$ such that $(T-\lambda)_{[d]} \in \Phi_{SF_+}(\mathcal{X})$, $\mathrm{ind}(T-\lambda)_{[d]} \leq 0$ and $(T-\lambda)_{[d]}$ has SVEP at 0. Hence $\lambda \in \mathrm{iso}\sigma_a(T)$. $\square$

Recall that a pair of operators $A, B \in B(\mathcal{X})$ is said to be *quasinilpotent equivalent* if

$$\lim_{n\longrightarrow\infty} ||\delta^n_{AB}(I)||^{\frac{1}{n}} = \lim_{n\longrightarrow\infty} ||\delta^n_{BA}(I)||^{\frac{1}{n}} = 0.$$

Here $\delta_{AB} \in B(B(\mathcal{X}))$ is the generalized derivation $\delta_{AB}(X) = AX - XB$ and $\delta^n_{AB}(X) = \delta^{n-1}_{AB}(\delta_{AB}(X))$.

Theorem 8.3.7 *$gBt \Longleftrightarrow Bt$ and $a - gBt \Longleftrightarrow a - Bt$.*

Proof. We have only to prove that $Bt \Longrightarrow gBt$ and $a - Bt \Longrightarrow a - gBt$. Let $T \in B(\mathcal{X})$. In view of Lemma 8.3.6, it would suffice to prove that T has SVEP at $\lambda \notin \sigma_w(T) \Longrightarrow T$ has SVEP at $\lambda \notin \sigma_{BW}(T)$ and T has SVEP at $\lambda \notin \sigma_{aw}(T) \Longrightarrow T$ has SVEP at $\lambda \notin \sigma_{SBW_+^-}(T)$. Suppose that $\lambda \notin \sigma_{BW}(T)$ (resp., $\lambda \notin \sigma_{SBW_+^-}(T)$). Then, for a large enough $n \in \mathbf{N}$, the operator $V_n = T - \lambda - \frac{1}{n} \in \Phi(\mathcal{X})$ with $\mathrm{ind}(V_n) = 0$ (resp., $V_n = T - \lambda - \frac{1}{n} \in \Phi_{SF_+^-}(\mathcal{X})$) [22, Theorem 4.7]. Equivalently, $\lambda - \frac{1}{n} \notin \sigma_w(T)$ (resp., $\lambda - \frac{1}{n} \notin \sigma_{aw}(T)$). It is clear that if T has SVEP at points $\mu \notin \sigma_w(T)$ (resp., $\mu \notin \sigma_{aw}(T)$), then V_n has SVEP at 0. Since T commutes with V_n, V_n converges to $T - \lambda$ in the uniform topology, and the operators V_n and $T - \lambda$ are quasinilpotent equivalent, [30, Proposition 3.4.11] implies that $T - \lambda$ has SVEP at 0. $\square$

Theorem 8.3.7 implies that the (equivalent) conditions of Theorem 8.3.1 are equivalent to the condition T satisfies gBt, and the (equivalent) conditions of Theorem 8.3.3 are equivalent to the condition T satisfies $a - gBt$. The following theorem lists some further equivalent conditions. Recall, [31], that an operator $T \in B(\mathcal{X})$ of topological uniform descent d such that $T^{d+1}\mathcal{X}$

is closed is said to be a *quasi–Fredholm operator of degree d*. Operators $T \in \Phi_{SBF_+^-}(\mathcal{X})$ are quasi–Fredholm (of some degree d). Let $\sigma_{BB}(T) = \{\lambda \in \sigma(T) : T - \lambda \notin \Phi_{BF}(\mathcal{X})$ or $\mathrm{asc}(T-\lambda) \neq \mathrm{dsc}(T-\lambda)\}$, and $\sigma_{SBB_+}(T) = \{\lambda \in \sigma_a(T) : T - \lambda \notin \Phi_{SBF_+}(\mathcal{X})$ or $\mathrm{asc}(T - \lambda = \infty\}$. It is easily verified that $\sigma_{BB}(T) = \sigma_{BW}(T)$ if and only if T has SVEP at points $\lambda \notin \sigma_{BW}(T)$; again, $\sigma_{SBB_+}(T) = \sigma_{SBW_+^-}(T)$ if and only if T has SVEP at points $\lambda \notin \sigma_{SBW_+^-}(T)$.

For an arbitrary closed subset F of $\mathbf{C}$, and $T \in B(\mathcal{X})$, let $X_T(F) = \{x \in \mathcal{X} : (T - \lambda)f_x(\lambda) \equiv x$ for some analytic function $f_x : \mathbf{C} \setminus F \longrightarrow \mathcal{X}\}$. The *glocal spectral subspace* $X_T(F)$ is a hyper-invariant linear manifold of T [30, p. 220] such that $X_T(\emptyset)$ is trivial and $X_T(\{0\}) = H_0(T)$ [2, p. 68].

Theorem 8.3.8 *For operators $T \in B(\mathcal{X})$, the following conditions are equivalent.*
(a). (i) T satisfies gBt (or Bt).
(ii) $f(T)$ satisfies gBt for every $f \in H_c(\sigma(T))$.
(iii) $\sigma(T) = \sigma_{BW}(T) \cup \partial\sigma(T)$ (where $\partial\sigma(T)$ denotes the boundary of $\sigma(T)$).
(iv) $H_0(T-\lambda) \cap K(T-\lambda) = \{0\}$ at points $\lambda \notin \sigma_{BW}(T)$.
(v) $\sigma(T) \setminus \sigma_{BW}(T) = \mathcal{R}(T)$.
(vi) $\sigma_{BB}(T) = \sigma_{BW}(T)$.
(b). (i) T satisfies $a - gBt$ (or $a - Bt$).
(ii) $f(T)$ satisfies $a - gBt$ for every $f \in H_c(\sigma(T))$.
(iii) $\sigma_a(T) = \sigma_{SBW_+^-}(T) \cup \partial\sigma_a(T)$.
(iv) there exists an $n \in \mathbf{N}$ such that $H_0(T-\lambda) = (T-\lambda)^{-n}(0)$ at points $\lambda \notin \sigma_{SBW_+^-}(T)$.

(v) T has SVEP at points $\lambda \in \sigma_a(T)$ for which $T - \lambda$ is quasi–Fredholm of degree d for some $d \in \mathbf{N}$.

(vi) $\sigma(T) \setminus \sigma_{SBW_+^-}(T) = \mathcal{R}^a(T)$.

(vii) $\sigma_{SBB_+}(T) = \sigma_{SBW_+^-}(T)$.

Proof. (a) (i)$\Longleftrightarrow$(ii). This follows from the observations that: T satisfies gBt if and only if T has SVEP at points $\lambda \notin \sigma_{BW}(T)$; if $f \in H_c(\sigma(T))$, then $f(T)$ has SVEP at points $\mu \notin \sigma_{BW}(f(T))$ if and only if T has SVEP at points $\lambda \notin \sigma_{BW}(T)$ such that $f(\lambda) = \mu$ (see Lemma 8.2.6).

(i)$\Longleftrightarrow$ (iii). Since T satisfies $gBt \iff \mathrm{acc}\sigma(T) \subseteq \sigma_{BW}(T)$, $\sigma(T) = \mathrm{acc}\sigma(T) \cup \mathrm{iso}\sigma(T) \subseteq \sigma_{BW}(T) \cup \mathrm{iso}\sigma(T) \subseteq \sigma_{BW}(T) \cup \partial\sigma(T) \subseteq \sigma(T)$. Again, if $\sigma(T)=\sigma_{BW}(T)\cup\partial\sigma(T)$, then T has SVEP at points not in $\sigma_{BW}(T)\Longrightarrow T$ satisfies gBt.

(i)$\Longleftrightarrow$(iv). If $\lambda\notin\sigma_{BW}(T)$, then (as observed above) $T-\lambda$ is Kato type [32, Theorem 7]. Hence, if T has SVEP at λ, then $\mathcal{X}=H_0(T-\lambda)\oplus K(T-\lambda)=(T-\lambda)^{-p}(0)\oplus(T-\lambda)^p\mathcal{X}$ for some $p\in\mathbf{N}$. Thus (i)$\Longrightarrow$ (iv). Conversely, since $(T-\lambda)^{-m}(0)\subseteq H_0(T-\lambda)$ for every $m\in\mathbf{N}$, operators $T\in B(\mathcal{X})$ and $\lambda\in\mathbf{C}$, if (iv) is satisfied, then $(T-\lambda)^{-1}(0)\cap K(T-\lambda)=\{0\}$. This, [2, Theorem 2.22], implies that T has SVEP at λ.

(i)$\Longleftrightarrow$(v) and (i)$\Longleftrightarrow$(vi) being evident, the proof is complete.

(b) The proof here, except for the case (i) $\Longleftrightarrow$(iv) and (i)$\Longleftrightarrow$(v), is similar to that above: we prove (i)$\Longleftrightarrow$(iv) and (i)$\Longleftrightarrow$(v).

i)$\Longleftrightarrow$(iv). Suppose that (i) holds. Assume without loss of generality that $\lambda=0$ ($\notin\sigma_{SBW_+^-}(T)$). Then there exists a $d\in\mathbf{N}$ such that $T^d\mathcal{X}$ is closed and $T_{[d]}$ is semi-regular. Let $\hat{T}_{[d]}:\mathcal{X}/T^{-d}(0)\longrightarrow\mathcal{X}/T^{-d}(0)$ denote the canonical quotient map, and define the operator $[T^d]:\mathcal{X}/T^{-d}(0)\longrightarrow T^d\mathcal{X}$ by setting $[T^d]x=T^dx$ for all $x\in\mathcal{X}/T^{-d}(0)$. Then $[T^d]$ is a bijection which satisfies $[T^d]\hat{T}_{[d]}=T_{[d]}[T^d]$. Since $T_{[d]}$ has SVEP at 0, $\hat{T}_{[d]}\in\Phi_{SF_+}$ has SVEP at 0; hence $\operatorname{asc}(\hat{T}_{[d]})=p<\infty$ for some $p\in\mathbf{N}$, which implies that $H_0(\hat{T}_{[d]})=(\hat{T}_{[d]})^{-p}(0)$. To complete the proof, we show that $x\in H_0(T)$ implies $\hat{x}\in H_0(\hat{T}_{[d]})$, where $\hat{x}$ denotes the equivalence class of $x\in\mathcal{X}/T^{-d}(0)$. Let $\mathbf{H}(\Omega,\mathcal{X})$ denote the Frèchet space of analytic functions from an open set $\Omega\subseteq\mathbf{C}$ into $\mathcal{X}$. Since $H_0(T)=X_T(\{0\})$, for every $x\in H_0(T)$ there exists $g\in\mathbf{H}(\mathbf{C}\setminus\{0\},\mathcal{X})$ such that $x=(T-\mu)g(\mu)$ for all $\mu\in\mathbf{C}\setminus\{0\}$. Letting $\phi:\mathcal{X}\longrightarrow\mathcal{X}/T^{-d}(0)$ denote the canonical quotient map, it follows that there exists $\hat{g}=\phi\circ g$ such that $\hat{x}=(\hat{T}_{[d]}-\mu)\hat{g}(\mu)$ for all $\mu\in\mathbf{C}\setminus\{0\}$. But then $\hat{x}\in\hat{X}_{\hat{T}_{[d]}}(\{0\})=H_0(\hat{T}_{[d]})=\hat{T}_{[d]}^{-p}(0)$ for some $p\in\mathbf{N}$. This implies that $T^px\in T^d(0)$, and hence that $H_0(T)\subseteq T^{-(p+d)}(0)$. Since $T^{-m}(0)\subseteq H_0(T)$ for every $m\in\mathbf{N}$, $H_0(T)=T^{-(p+d)}(0)$. To complete the proof, assume now that $H_0(T-\lambda)=(T-\lambda)^{-n}(0)$, for some $n\in\mathbf{N}$, at every $\lambda\notin\sigma_{SBW_+^-}(T)$. Then $H_0(T-\lambda)$ is closed, which implies that T has SVEP at λ.

(i)$\Longleftrightarrow$(v). If we let S denote the set of $\lambda\in\sigma_a(T)$ such that T has SVEP at λ and $T-\lambda$ is quasi–Fredholm of degree d for some $d\in\mathbf{N}$, and $G=\{\lambda\in\sigma_a(T):\lambda\notin\sigma_{SBW_+^-}(T)$ and T has SVEP at $\lambda\}$, then $G\subseteq S$. The argument of the proof of Lemma 8.2.2 implies that $\lambda \in S \Longrightarrow \lambda \in \Phi_{SBF_+^-}(T)$ such that T has SVEP at λ. Hence $S \subseteq G$, which implies that $S = G$. Since (i) is satisfied if and only if T has SVEP at points λ such that $T - \lambda \in \Phi_{SBF_+^-}(\mathcal{X})$, the proof follows. $\square$

Remark 8.3.9 It is clear from the example following Corollary 8.3.4 and Theorem 8.3.7 that the implications $gBt \Longrightarrow a - gBt$ fails. Also, T satisfies $a - gBt$ does not imply that T^* satisfies $a - gBt$. (Observe that $\sigma_{BW}(T) = \sigma_{BW}(T^*)$; hence T satisfies gBt if and only if T^* satisfies gBt.) It is apparent from Corollary 8.3.5 that if either of T and T^* has SVEP, then both T and T^* satisfy $a - gBt$. Furthermore, in such a case, either $f(T)$ or $f(T^*)$ has SVEP for every $f \in H(\sigma(T))$; consequently, $f(T)$ and $f(T^*)$ satisfy $a - gBt$, hence also gBt, $a - Bt$ and Bt.

Perturbations. If $T \in B(\mathcal{X})$ and $Q \in B(\mathcal{X})$ is a quasinilpotent operator which commutes with T, then T and $T+Q$ are quasinilpotent equivalent. This [30, Theorem 3.4.9] implies that $T+Q$ has SVEP at a point if and only if T has SVEP at the point. Hence:

Theorem 8.3.10 *If T and Q are as above, then $T + Q$ satisfies $a - gBt$ (or gBt or $a - Bt$ or Bt) if and only if T does.*

Now let $K \in B(\mathcal{X})$ be a compact operator which commutes with T. Then $\sigma_x(T + K) = \sigma_x(T)$, where σ_x stands for either of σ_b, σ_w, σ_{ab} and σ_{aw} [2, p. 145, 151]. Hence:

Theorem 8.3.11 *If $K \in B(\mathcal{X})$ is a compact operator which commutes with $T \in B(\mathcal{X})$, then $T + K$ satisfies $a - gBt$ (or gBt or $a - Bt$ or Bt) if and only if T does.*

The commutativity hypothesis in the above theorems is essential, as follows from a consideration of the operator $T = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} + \begin{pmatrix} 0 & 1 - UU^* \\ 0 & 0 \end{pmatrix}$, $U \in B(\mathcal{H})$ the forward unilateral shift, when it is seen that T (being unitary) satisfies $a - gBt$ but $U \oplus U^*$ does not satisfy Bt (hence, $a - Bt$ or $g - Bt$

or $a - gBt$). Notice that T has SVEP, but $U \oplus U^*$ does not have SVEP at points not in $\sigma_w(U \oplus U^*)$.

Every compact operator is a Riesz operator, where $R \in B(\mathcal{X})$ is said to be a *Riesz operator* if its essential spectral radius equals 0. It is known, see for example [35] and [15], that if R is a Riesz operator which commutes with T, then $\sigma_x(T + R) = \sigma_x(T)$, where $\sigma_x = \sigma_{ab}$ or σ_{sb} or σ_{aw} or σ_{sw} (consequently, also σ_b and σ_w).

Theorem 8.3.12 *If $R \in B(\mathcal{X})$ is a Riesz operator which commutes with $T \in B(\mathcal{X})$, then $T + R$ (resp., $T^* + R^*$) satisfies $a - gBt$ (or gBt or $a - Bt$ or Bt) if and only if T (resp., T^*) does.*

Proof. . Since $\sigma_{ab}(T+R) = \sigma_{ab}(T)$ and $\sigma_{aw}(T+R) = \sigma_{aw}(T)$, $\sigma_{ab}(T+R) = \sigma_{aw}(T + R)$ if and only if $\sigma_{ab}(T) = \sigma_{aw}(T)$, i.e. $T + R$ satisfies $a - Bt$ (equivalently, $a-gBt$) if and only if T satisfies $a-Bt$ (equivalently, $a-gBt$). Again, since $\sigma_{sb}(T+R) = \sigma_{sw}(T+R)$ if and only if $\sigma_{sb}(T) = \sigma_{sw}(T)$ implies $\sigma_{ab}(T^* + R^*) = \sigma_{aw}(T^* + R^*)$ if and only if $\sigma_{ab}(T^*) = \sigma_{aw}(T^*)$, $T^* + R^*$ satisfies $a - Bt$ (equivalently, $a - gBt$) if and only if T^* satisfies $a - Bt$ (equivalently, $a - gBt$). □

We consider next perturbations by *algebraic operators*, i.e., operator $A \in B(\mathcal{X})$ for which there exists a (non-constant) polynomial $p(.)$ such that $p(A) = 0$. (Operators $F \in B(\mathcal{X})$ such that F^n is finite dimensional for some $n \in \mathbf{N}$ are algebraic.)

If $A \in B(\mathcal{X})$ is an algebraic operator, then $\sigma(A) = \{\mu_1, \mu_2, ..., \mu_n\}$ for some scalars μ_i, $1 \leq i \leq n$ (for some $n \in \mathbf{N}$). Suppose that $T \in B(\mathcal{X})$ commutes with A. Let $T_i = T|_{H_0(A_i-\mu_i)}$ and $A_i = A|_{H_0(A_i-\mu_i)}$. Since the projection $H_0(A_i - \mu_i)$ corresponding to μ_i commutes with T for all $1 \leq i \leq n$, $T = \bigoplus_{i=1}^n T_i$ and $T+A = \bigoplus_{i=1}^n (T_i+A_i)$. Evidently, $\sigma(A_i) = \{\mu_i\}$ and $p(A_i) = 0$ for some polynomial $p(.)$ and all $1 \leq i \leq n$. Since $\sigma(p(A_i)) = p(\sigma(A_i)) = p(\mu_i) = 0$, $0 = p(A_i) = p(A_i)-p(\mu_i) = (A_i-\mu_i)^{m_i} g(A_i)$ for some $m_i \in \mathbf{N}$ and invertible $g(A_i)$. Hence $A_i - \mu_i$ is nilpotent, and the operators $T_i + A_i - \mu_i$ and T are quasinilpotent equivalent. Consequently, $T_i + A_i - \mu_i$ has SVEP at a point if and only if T_i has SVEP at the point.

Theorem 8.3.13 *Let $A \in B(\mathcal{X})$ be an algebraic operator which commutes with an operator $T \in B(\mathcal{X})$. If T has SVEP, then $T + A$ and $T^* + A^*$ satisfy $a - gBt$.*

Proof. Since a diagonal operator has SVEP if and only if each of its entries has SVEP, it follows that $T + A = \bigoplus_{i=1}^{n}(T_i + A_i)$ has SVEP. This implies that both $T + A$ and $T^* + A^*$ satisfy $a - gBt$. $\square$

8.4. Weyl's Theorem

For an operator $T \in B(\mathcal{X})$, let

$$\begin{aligned}
\Pi(T) &= \{\lambda \in \mathrm{iso}\sigma(T) : 0 < \alpha(T - \lambda)\};\\
\Pi^a(T) &= \{\lambda \in \mathrm{iso}\sigma_a(T) : 0 < \alpha(T - \lambda)\};\\
\Pi_0(T) &= \{\lambda \in \Pi(T) : \alpha(T - \lambda) < \infty\};\\
\Pi_0^a(T) &= \{\lambda \in \Pi^a(T) : \alpha(T - \lambda) < \infty\}.
\end{aligned}$$

We say that T satisfies:

$$\begin{aligned}
&\text{Weyl's theorem, or } Wt, \text{if } \sigma(T) \setminus \sigma_w(T) = \Pi_0(T);\\
&a - \text{Weyl's theorem, or } a - Wt, \text{if } \sigma_a(T) \setminus \sigma_{aw}(T) = \Pi_0^a(T);\\
&\text{generalized Weyl's theorem, or } gWt, \text{if } \sigma(T) \setminus \sigma_{BW}(T) = \Pi(T);\\
&\text{generalized } a - \text{Weyl's theorem, or } a - gWt, \text{if}\\
&\qquad\qquad \sigma_a(T) \setminus \sigma_{SBW_+^-}(T) = \Pi^a(T).
\end{aligned}$$

Theorem 8.4.1 *The following implications hold:*
(i) $Wt \Longrightarrow Bt$, $a - Wt \Longrightarrow a - Bt$, $gWt \Longrightarrow gBt$, and $a - gWt \Longrightarrow a - gBt$;
(ii) $a - gWt \Longrightarrow gWt \Longrightarrow Wt$;
(iii) $a - gWt \Longrightarrow a - Wt \Longrightarrow Wt$.

Proof. Let $T \in B(\mathcal{X})$.
(i) If $\lambda \notin \sigma_w(T)$ or $\sigma_{BW}(T)$ (resp., $\lambda \notin \sigma_{aw}(T)$ or $\sigma_{SBW_+^-}(T)$), then $\lambda \in \mathrm{iso}\sigma(T)$ (resp., $\lambda \in \mathrm{iso}\sigma_a(T)$). In either case T has SVEP at all such λ, which proves (i).
(ii) If T satisfies $a - gWt$, then T satisfies $gBt \Longrightarrow \sigma(T) \setminus \sigma_{BW}(T) = \mathcal{R}(T) \subseteq$

$\Pi(T)$. Let $\lambda \in \Pi(T)$ ($\subseteq \Pi^a(T) = \sigma_a(T) \setminus \sigma_{SBW_+^-}(T)$). Then $\lambda \notin \sigma_{SBW_+^-}(T)$. Since $\lambda \in \text{iso}\sigma(T)$, T^* has SVEP at λ. But then $T - \lambda \in \Phi_{BF}(\mathcal{X})$ and $\text{asc}(T-\lambda) = \text{dsc}(T-\lambda) < \infty$. It follows that $\mathcal{R}(T) = \Pi(T)$. Thus T satisfies gWt. The proof of the remaining implication is similar.

(iii) If T satisfies $a-gWt$, then T satisfies $a-gBt$, hence also $a-Bt \Longleftrightarrow \sigma_a(T) \setminus \sigma_{aw}(T) = \mathcal{R}_0^a(T) \subseteq \Pi_0^a(T)$. Now let $\lambda \in \Pi_0^a(T)$. Since $\Pi_0^a(T) \subseteq \Pi^a(T)$ ($= \sigma_a(T) \setminus \sigma_{SBW_+^-}(T)$), and since T satisfies $a-gWt$), $\lambda \notin \sigma_{SBW_+^-}(T)$. The hypothesis T satisfies $a-gWt$ thus implies that $\lambda \in \mathcal{R}^a(T)$. Observe that if $\alpha(T-\lambda) < \infty$, then $T - \lambda \in \Phi_{SF_+^-}(\mathcal{X})$ and $\lambda \in \mathcal{R}_0^a(T)$. Hence, T satisfies $a - Wt$. The proof of the implication $aWt \Longrightarrow Wt$ is similar. $\square$

The implications of Theorem 8.4.1 do not reverse. Plenty of examples proving this are to be found in the extant literature: we shall here be content with the following examples.

Example 8.4.2 Let $Q \in \ell^2(\mathbf{N})$ be the quasi–nilpotent $Q(x_0, x_1, x_2, ...) = (\frac{1}{2}x_1, \frac{1}{3}x_2, ...)$ and $N \in B(\ell^2)$ be a nilpotent. Let $T = Q \oplus N$. Then $\sigma_W(T) = \sigma_{BW}(T) = \sigma_{SBW_+^-}(T) = \{0\} = \sigma(T) = \sigma_a(T) = E(T) = E^a(T)$ and $E_0(T) = E_0^a(T) = \emptyset$, which implies that T satisfies $a - Wt$ (so also Wt) but fails to satisfy gWt (so also $a - gWt$). Again, let $S \in \ell^2(\mathbf{N})$ denote the weighted forward unilateral shift with the weight sequence $\{\frac{1}{n+1}\}$. Then S is an injective quasinilpotent such that the range of S^m is not closed for every $m \in \mathbf{N}$. Let $Q = S \oplus S$, and define $A \in \ell^2(\mathbf{N}) \oplus \ell^2(\mathbf{N})$ by $A = (I - S) \oplus 0$. Then A commutes with S. Now set $T = A + Q$. Then $\sigma(T) = \sigma_w(T) = \sigma_{BW}(T) = \{0, 1\} = \sigma_a(T^*) = \sigma_{SBW_+^-}(T^*)$, $\Pi_0(T) = \emptyset$, $\Pi(T) = \{1\}$, $\Pi^a(T^*) = \{1\}$, and $\mathcal{R}(T) = \mathcal{R}(T^*) = \emptyset$. Evidently, T satisfies Wt, both T and T^* satisfy $a - gBt$, but both T and T^* do not satisfy gWt or $a - gWt$.

We shall say in the following that an operator $T \in B(\mathcal{X})$ is *polaroid on a set* $F \subseteq iso\sigma(T)$ (resp., *a-polaroid on a set* $F \subseteq iso\sigma_a(T)$) if $\text{asc}(T-\lambda) = \text{dsc}(T-\lambda) < \infty$ for every $\lambda \in F$ (resp., $\text{asc}(T-\lambda) < \infty$ and $(T-\lambda)\mathcal{X}$ is closed for every $\lambda \in F$); T is *left polaroid on* $F \subseteq iso\sigma_a(T)$ if every $\lambda \in F$ is a left pole of T. T would (simply) be said to be *polaroid* (or *a–polaroid, left polaroid*) if $F = \text{iso}\sigma(T)$ (resp., $F = \text{iso}\sigma_a(T)$).

The following theorem characterizes operators T satisfying Weyl's theorem.

Theorem 8.4.3 *$T \in B(\mathcal{X})$ satisfies:*
(i) Wt if and only if T has SVEP at points $\lambda \notin \sigma_w(T)$ and T is polaroid on the set $\Pi_0(T)$.
(ii) $a - Wt$ if and only if T has SVEP at points $\lambda \notin \sigma_{aw}(T)$ and T is a-polaroid on the set $\Pi_0^a(T)$.
(iii) gWt if and only if T has SVEP at points $\lambda \notin \sigma_{BW}(T)$ and T is polaroid on the set $\Pi(T)$.
(iv) $a - gWt$ if and only if T has SVEP at points $\lambda \notin \sigma_{SBW_+^-}(T)$ and T is left polaroid on the set $\Pi^a(T)$.

Proof. The proof in all the cases is similar: we prove (iv). If T satisfies $a-gWt$, then (as already seen) T satisfies $a-gBt$. Hence $\sigma(T)\setminus\sigma_{SBW_+^-}(T) = \mathcal{R}^a(T) = \Pi^a(T)$. Conversely, if T satisfies $a - gBt$, i.e. $\sigma(T) \setminus \sigma_{SBW_+^-}(T) = \mathcal{R}^a(T)$, and if $\Pi^a(T) \subseteq \mathcal{R}^a(T)$, then $\Pi^a(T) = \mathcal{R}^a(T)$ and T satisfies $a - gWt$. $\square$

Although the reverse of the implications $a-Wt \Longrightarrow Wt$ and $a-gWt \Longrightarrow gWt$ does not in general hold, there are classes of operators for which these one sided implications are an equivalence. One such class of operators is that of operators T for which T^* has SVEP. Observe that if T^* has SVEP, then

$$\sigma(T) = \sigma_a(T), \sigma_w(T) = \sigma_{aw}(T),\ \ \sigma_{BW}(T) = \sigma_{SBW_+^-}(T), \Pi_0(T) = \Pi_0^a(T)$$

and

$$\Pi(T) = \Pi^a(T).$$

Hence, *if T^* has SVEP, then T satisfies Wt (resp., gWt) if and only if T satisfies $a - Wt$ (resp., $a - gWt$).* What happens if T has SVEP? In this case the implications are liable to fail: consider, for example the forward unilateral shift $S \in B(\mathcal{H})$ (which has SVEP, satisfies Wt but does not satisfy $a - Wt$). However, not all is lost if one assumes T to be polaroid.

Proposition 8.4.4 *Let $T \in B(\mathcal{X})$ be a polaroid operator such that T has SVEP. Then $f(T)$ satisfies gWt for every $f \in H(\sigma(T))$ and $f(T)^*$ satisfies $a - gWt$ for every $f \in H_c(\sigma(T))$.*

Proof. The hypothesis T has SVEP implies that $\sigma(T)(= \sigma(T^*)) = \sigma_a(T^*)$, $\Pi^a(T^*) = \Pi(T)$, $\sigma_{SBW_+^-}(T^*) = \sigma_{BW}(T) = \sigma_{BW}(T^*)$, $\sigma(T) \setminus \sigma_{BW}(T) = \mathcal{P}(T)$ and $\sigma_a(T^*) \setminus \sigma_{SBW_+^-}(T^*) = \mathcal{P}^a(T^*)$. Evidently, $\mathcal{P}(T) \subseteq \Pi(T)$ and $\mathcal{P}^a(T^*) \subseteq \Pi^a(T^*)$. Since $\lambda \in \Pi(T)$ implies $\lambda \in \mathrm{iso}\sigma(T)$, and since T is polaroid, $\lambda \in \Pi(T) \Longrightarrow \lambda \in \mathcal{P}(T)$. Hence, T satisfies $g - Wt$. Again, $\lambda \in \Pi^a(T^*) \Longrightarrow \lambda \in \mathrm{iso}\sigma(T^*) = \mathrm{iso}\sigma(T)$. Since T is polaroid, $\lambda \in \Pi^a(T^*) \Longrightarrow \lambda \in \mathcal{P}(T)$. Recall now the (easily verified) fact that T is polaroid at a point implies that T^* is polaroid at the point; hence $\lambda \in \Pi^a(T^*) \Longrightarrow \mathcal{P}(T) \subseteq \mathcal{P}(T^*) = \mathcal{P}^a(T^*)$, which implies that T^* satisfies $a - gWt$.

Recall that an operator $A \in B(\mathcal{X})$ is said to be *isoloid* (resp., *a–isoloid*) if $\{\lambda \in \mathrm{iso}\sigma(A)\} \subseteq \Pi(A)$ (resp., $\{\lambda \in \mathrm{iso}\sigma_a(A)\} \subseteq \Pi^a(A)$). Apparently, the operator T being polaroid is isoloid. The following argument shows that $f(\sigma_{BW}(T)) = f(\sigma(T) \setminus \Pi(T)) = \sigma(f(T)) \setminus \Pi(f(T))$; this, since $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$ for every $f \in H(\sigma(T))$, would then imply that $f(T)$ satisfies gWt. If $\lambda \in \sigma(f(T)) \cap \Pi(f(T))$, then there exists a $\mu \in \mathrm{iso}\sigma(T)$ such that $f(\mu) = \lambda$, and then (by the isoloid property) $\mu \in \Pi(T) \Longrightarrow \sigma(f(T)) \setminus \Pi(f(T)) \supseteq f(\sigma(T) \setminus \Pi(T))$. For the reverse inclusion, let $\lambda \in \sigma(f(T)) \setminus \Pi(f(T))$. If $\lambda \in \mathrm{iso}\sigma(f(T))$, and $\lambda \notin \Pi(f(T))$, then $f(T) - \lambda = c_0 \Pi_{i=1}^n (T - \mu_i) g(T)$ for some scalars c_0 and μ_i $(1 \leq i \leq n)$ and invertible operator $g(T)$. Since $\mu_i \notin \Pi(T)$ for all $1 \leq i \leq n$, $\lambda \in f(\sigma(T) \setminus \Pi(T))$. If, on the other hand, $\lambda \notin \mathrm{iso}\sigma(f(T))$, then there exists a sequence $\{\lambda_n\}$ in $\sigma(f(T))$ and a sequence $\{\mu_n\}$ in $\sigma(T)$ such that $f(\mu_n) = \lambda_n$ converges to λ. $\sigma(T)$ being a compact subset of $\mathbf{C}$, we may assume that $\mu_n \longrightarrow \mu \in \sigma(T)$, and then $\lambda = g(\mu) \in f(\sigma(T) \setminus \Pi(T))$.

Again, since $\sigma_a(T^*) = \sigma(T^*) = \sigma(T)$, T^* is a–isoloid (indeed, $\Pi(T) = \Pi^a(T^*)$), which (see above) implies that $f(\sigma_a(T^*) \setminus \Pi^a(T^*)) = f(\sigma_a(T^*)) \setminus f(\Pi^a(T^*))$. This, since $f(\sigma_{SBW_+^-}(T^*)) = f(\sigma_{BW}(T^*)) = \sigma_{BW}(f(T^*)) = \sigma_{SBW_+^-}(f(T^*))$ (see Lemma 8.2.8), implies that $f(T^*)$ satisfies $a - gWt$ for every $f \in H_c(\sigma(T))$. $\square$

Just as SVEP at points $\lambda \notin \sigma_w(T)$ (or $\sigma_{aw}(T)$ or $\sigma_{BW}(T)$ or $\sigma_{SBW_+^-}(T)$) ensuring Bt (resp., $a-Bt$ or gBt or $a-gBt$) may be replaced by a number of equivalent conditions (see Section 3), the polaroid (resp., a-polaroid or left polaroid) property in Theorem 8.4.3 may be replaced by a number of equivalent conditions. We list below some of these conditions – many others are to be found in the literature.

Theorem 8.4.5 *$T \in B(\mathcal{X})$ satisfies Wt if and only if T satisfies Bt and one of the following equivalent conditions.*
(i) $\Pi_0(T) \subseteq \mathcal{R}_0(T)$.
(ii) $T-\lambda$ is Kato type at points $\lambda \in \Pi_0(T)$.
(iii) $(T-\lambda)\mathcal{X}$ is closed at points $\lambda \in \Pi_0(T)$.
(iv) $\dim H_0(T-\lambda) < \infty$ *at points $\lambda \in \Pi_0(T)$.*
(v) co-dim$K(T-\lambda) < \infty$ at points $\lambda \in \Pi_0(T)$.
(vi) $H_0(T-\lambda) = (T-\lambda)^{-p}(0)$ for some $p \in \mathbf{N}$ at points $\lambda \in \Pi_0(T)$.
(vii) $K(T-\lambda) = (T-\lambda)^p\mathcal{X}$ for some $p \in \mathbf{N}$ at points $\lambda \in \Pi_0(T)$.
(viii) dsc$(T-\lambda) < \infty$ at points $\lambda \in \Pi_0(T)$.
(ix) The mapping $\lambda \to \gamma(T-\lambda)$ is not continuous at points $\lambda \in \Pi_0(T)$.

Proof. Let $\lambda \in \Pi_0(T)$. Then $\lambda \in \text{iso}\sigma(T) \Longrightarrow \mathcal{X} = H_0(T-\lambda) \oplus K(T-\lambda)$. (i)$\Longrightarrow$ (ii) is evident. If (ii) holds, then $H_0(T-\lambda) = (T-\lambda)^{-p}(0)$ for some $p \in \mathbf{N} \Longrightarrow (T-\lambda)^p\mathcal{X} = 0 \oplus (T-\lambda)^p K(T-\lambda) = K(T-\lambda) \Longrightarrow \mathcal{X} = (T-\lambda)^{-p}(0) \oplus (T-\lambda)^p\mathcal{X} \Longrightarrow \text{asc}(T-\lambda) = \text{dsc}(T-\lambda) < \infty$. Hence (ii)$\Longrightarrow$ (i), and (ii) $\Longrightarrow$ (vi) $\Longrightarrow$ (vii) $\Longrightarrow$ (viii). Again, if (ii) holds, then $\alpha(T-\lambda) = \beta(T-\lambda) < \infty \Longrightarrow (T-\lambda)\mathcal{X}$ is closed, so that (ii) $\Longrightarrow$ (iii). Observe that if $(T-\lambda)\mathcal{X}$ is closed and $\alpha(T-\lambda) < \infty$, then $T-\lambda \in \Phi_+(\mathcal{X})$. This, if $\lambda \in \text{iso}\sigma(T)$, then implies that $H_0(T-\lambda) = (T-\lambda)^{-p}(0)$ for some $p \in \mathbf{N}$. Consequently, (iii) $\Longrightarrow$ (iv). Evidently, co-dim$K(T-\lambda) = \dim H_0(T-\lambda)$; hence (iv) $\Longrightarrow$ (v). Since $K(T-\lambda) \subseteq \cap_{n \in \mathbf{N}}(T-\lambda)^n\mathcal{X} \subseteq (T-\lambda)\mathcal{X}$, (v) implies that $\beta(T-\lambda) < \infty$. Since already $\alpha(T-\lambda) < \infty$, (v) implies $T-\lambda \in \Phi(\mathcal{X})$; hence, because both T and T^* have SVEP at λ, $\text{asc}(T-\lambda) = \text{dsc}(T-\lambda) < \infty$, i.e. (v) $\Longrightarrow$ (i). To complete the proof, we now prove that (ix) $\Longleftrightarrow$ (viii) $\Longrightarrow$ (i). It is easily seen that $\gamma(T)$ is not continuous at λ if and only if $(T-\lambda)\mathcal{X}$ is closed; hence (ix)$\Longleftrightarrow$ (iii)$\Longleftrightarrow$(i). If $\text{dsc}(T-\lambda) < \infty$, then $\beta(T-\lambda) \leq \alpha(T-\lambda) < \infty \Longrightarrow (T-\lambda)\mathcal{X}$ is closed; thus (viii) $\Longleftrightarrow$ (ix).

Finally, if (viii) holds, then $T-\lambda \in \Phi(\mathcal{X})$, and both T and T^* have SVEP. Consequently, (viii) $\Longrightarrow$ (i). $\square$

The following theorem is the analog of Theorem 8.4.5 for operators satisfying gWt.

Theorem 8.4.6 *$T \in B(\mathcal{X})$ satisfies gWt if and only if T satisfies gBt and one of the following equivalent conditions.*
(i) $\Pi(T) \subseteq \mathcal{R}(T)$.
(ii) $T-\lambda$ is Kato type at points $\lambda \in \Pi(T)$.
(iii) There exists an $n \in \mathbf{N}$ such that $H_0(T-\lambda) = (T-\lambda)^{-n}(0)$ at points $\lambda \in \Pi(T)$.
(iv) There exists an $n \in \mathbf{N}$ such that $(T-\lambda)^n\mathcal{X}$ is closed at points $\lambda \in \Pi(T)$.
(v) There exists an $n \in \mathbf{N}$ such that $K(T-\lambda) = (T-\lambda)^n\mathcal{X}$ at points $\lambda \in \Pi(T)$.
(vi) $dsc(T-\lambda) < \infty$ at points $\lambda \in \Pi(T)$.
(vii) There exists an $n \in \mathbf{N}$ such that the mapping $\lambda \rightarrow \gamma(T-\lambda)_{[n]}$ is not continuous at points $\lambda \in \Pi(T)$.
(viii) $T-\lambda$ has uniform topological descent at points $\lambda \in \Pi(T)$.

Proof. That (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$ (iv)$\Longrightarrow$(v)$\Longrightarrow$(vi) follows from an argument similar to that of the proof of Theorem 8.4.5. If (vi) holds, then $T-\lambda$ is of uniform topological descent for every $\lambda \in \Pi(T)$; hence (vi)$\Longrightarrow$ (viii). Recall from Grabiner [22, Corollary 4.9] that points λ in the boundary of $\sigma(T)$ such that $T-\lambda$ has uniform topological descent are poles of the resolvent. Thus (viii)$\Longrightarrow$ (i). Finally since $\gamma(T-\lambda)_{[n]}$ is not continuous at λ if and only if $(T-\lambda)^n\mathcal{X}$ is closed, (vii)$\Longleftrightarrow$(iv). $\square$

Our next result is an analogue of Theorems 8.4.5 and 8.4.6 for operator satisfying $a-Wt$ or $a-gWt$.

Theorem 8.4.7 *(a). If $T \in B(\mathcal{X})$ satisfies $a-Bt$, then the following conditions are equivalent: (i) T satisfies $a-Wt$.*
(ii) $\Pi_0^a(T) \subseteq \mathcal{R}_0^a(T)$.
(iii) $\sigma_{ab}(T) \cap \Pi_0^a(T) = \emptyset$.
(iv) $\sigma_{SF_+}(T) \cap \Pi_0^a(T) = \emptyset$.
(v) $(T-\lambda)\mathcal{X}$ is closed at points $\lambda \in \Pi_0^a(T)$.

(vi) $\dim(H_0(T-\lambda)) < \infty$ *at points* $\lambda \in \Pi_0(T)$ *and* $(T-\lambda)\mathcal{X}$ *is closed at points* $\lambda \in \Pi_0^a(T) \setminus \Pi_0(T)$.
(vii) $dsc(T-\lambda) < \infty$ *at points* $\lambda \in \Pi_0(T)$ *and* $(T-\lambda)\mathcal{X}$ *is closed at points* $\lambda \in \Pi_0^a(T) \setminus \Pi_0(T)$.
(viii) The mapping $\lambda \longrightarrow \gamma(T-\lambda)$ *is not continuous on* $\Pi_0^a(T)$.
(b). If $T \in B(\mathcal{X})$ *satisfies* $a-gBt$*, then the following conditions are equivalent:*
(i) T *satisfies* $a-gWt$.
(ii) $\Pi^a(T) \subseteq \mathcal{P}^a(T)$.
(iii) $H_0(T-\lambda) = (T-\lambda)^{-p}(0)$ *for some* $p \in \mathbf{N}$ *and* $(T-\lambda)^n\mathcal{X}$ *is closed for all* $n \geq p$ *and all* $\lambda \in \Pi^a(T)$.
(iv) $\lambda \notin \sigma_{LD}(T)$ *for every* $\lambda \in \Pi^a(T)$.
Furthermore, if $\mathcal{X} = \mathcal{H}$ *is a Hilbert space, then these conditions are equivalent to:*
(v) $T-\lambda$ *is Kato type at points* $\lambda \in \Pi^a(T)$.

Proof. (a). (i)$\Longleftrightarrow$ (ii) Evident (since an operator T satisfying $a-Bt$ satisfies $a-Wt$ if and only if $\Pi_0^a(T) = \mathcal{R}_0^a(T)$).
(ii)$\Longrightarrow$ (iii) $\Longrightarrow$ (iv)$\Longrightarrow$ (v). Since T satisfies $a-Bt \Longleftrightarrow \sigma_{ab}(T) = \sigma_{aw}(T) \Longleftrightarrow \sigma_a(T \setminus \sigma_{aw}(T) = \Pi_0^a(T)$, it follows from (ii) that $\sigma_{ab}(T) \cap \Pi_0^a(T) = \emptyset$. That (iii) implies (iv) is immediate from the fact that $\sigma_{SF_+}(T) \subseteq \sigma_{ab}(T)$. If (iv) holds, then $\lambda \in \Pi_0^a(T)$ implies that $\lambda \notin \sigma_{SF_+}(T)$, and hence that $(T-\lambda)\mathcal{X}$ is closed at points $\lambda \in \Pi_0^a(T)$.
(v)$\Longrightarrow$ (vi). If $(T-\lambda)\mathcal{X}$ is closed at points in $\Pi_0^a(T)$, then $T-\lambda \in \Phi_{SF_+}(\mathcal{X})$. Hence $T-\lambda$ is Kato type at points $\lambda \in \Pi_0(T)$ $(\subseteq \Pi_0^a(T))$. This implies that $H_0(T-\lambda) = (T-\lambda)^{-n}(0)$ for some $n \in \mathbf{N}$. Since $\alpha(T-\lambda) < \infty$, $\dim(H_0(T-\lambda)) < \infty$.
(vi)$\Longrightarrow$ (vii). If (vi) holds, then $\lambda \in \Pi_0(T) \Longrightarrow \mathcal{X} = H_0(T-\lambda) \oplus K(T-\lambda)$ with $\dim H_0(T-\lambda) < \infty$. Hence co-dim$K(T-\lambda) < \infty$. Since $K(T-\lambda) \subseteq (T-\lambda)\mathcal{X}$ (for every operator T), co-dim$(T-\lambda)\mathcal{X} = \beta(T-\lambda) < \infty$. Thus $T-\lambda \in \Phi(\mathcal{X})$ for all $\lambda \in \Pi_0(T)$. Since such a $\lambda \in \text{iso}\sigma(T)$, T^* has SVEP at λ. Hence dsc$(T-\lambda) < \infty$ at points $\lambda \in \Pi_0(T)$.
(vii)$\Longrightarrow$ (viii). If dsc$(T-\lambda) < \infty$ at points $\lambda \in \Pi_0(T)$, then $\beta(T-\lambda) \leq \alpha(T-\lambda) < \infty$. Hence $(T-\lambda)\mathcal{X}$ is closed at every $\lambda \in \Pi_0^a(T)$. Since (viii) is satisfied if and only if $(T-\lambda)\mathcal{X}$ is closed, the proof follows.

(viii)$\Longrightarrow$ (i). If (viii) holds, then $T-\lambda \in \Phi_{SF_+}(\mathcal{X})$ and (since T has SVEP at λ) $\text{ind}(T-\lambda) \leq 0$ at points $\lambda \in \Pi_0^a(T)$. Hence $\lambda \in \sigma_a(T) \setminus \sigma_{aw}(T)$ for all $\lambda \in \Pi_0^a(T)$. Thus $\Pi_0^a(T) \subseteq \sigma_a(T) \setminus \sigma_{aw}(T)$. Since already $\sigma_a(T) \setminus \sigma_{aw}(T) \subseteq \Pi_0^a(T)$ by the $a-Bt$ hypothesis, the proof is complete.

(b). (i)$\Longleftrightarrow$ (ii). Evident ((since an operator T satisfying $a-gBt$ satisfies $a-gWt$ if and only if $\Pi^a(T) = \mathcal{P}^a(T)$).
(ii)$\Longleftrightarrow$ (iii). If (ii) holds, then $\lambda \in \Pi^a(T)$ implies $\lambda \in \mathcal{P}^a(T)$, which by Lemma 8.2.2 implies that $\lambda \notin \sigma_{SBW_+^-}(T)$. Hence there exists an integer $p \in \mathbf{N}$ such that $(T-\lambda)^n\mathcal{X}$ is closed for all $n \geq p$. Argue as in the proof (i)$\Longrightarrow$ (iv) of Theorem 8.3.8 to prove that there exists a $p \in \mathbf{N}$ such that $H_0(T-\lambda) = (T-\lambda)^{-p}(0)$. Conversely, if (iii) holds, then $\lambda \in \Pi^a(T)$ implies $\text{asc}(T-\lambda) = p < \infty$, which in turn implies that $\text{ind}(T-\lambda) \leq 0$. Since already $(T-\lambda)^p\mathcal{X}$ is closed, $\lambda \notin \sigma_{SBW_+^-}(T) \Longrightarrow \lambda \in \mathcal{P}^a(T)$.
(ii)$\Longleftrightarrow$ (iv). If (ii) is satisfied, then $\lambda \in \Pi^a(T)$ implies $\lambda \notin \sigma_{SBW_+^-}(T)$. Since T has SVEP at λ, $\lambda \notin \sigma_{LD}(T)$. Conversely, if $\lambda \in \Pi^a(T)$ and $\lambda \notin \sigma_{LD}(T)$, then $\text{asc}(T-\lambda) = d < \infty$ and $(T-\lambda)^{d+1}\mathcal{X}$ is closed for some $d \in \mathbf{N} \Longrightarrow \lambda \in \mathcal{P}^a(T)$.

To complete the proof of the theorem, we start by proving that a sufficient condition for an operator $T \in B(\mathcal{X})$ satisfying $a-gBt$ to satisfy $a-gWt$ is that there exist an integer n such that T is Kato type of order n at points $\lambda \in \Pi^a(T)$. In view of the hypothesis T satisfies $a-gBt$, it would suffice to prove that $\Pi^a(T) \subseteq \mathcal{P}^a(T)$. Let $\lambda \in \Pi^a(T)$. Then T has SVEP at λ; hence, if T is Kato type at λ, then $\text{asc}(T-\lambda) = n < \infty$ for some $n \in \mathbf{N}$ [2, Theorems 3.23 and 3.16] and $(T-\lambda)\mathcal{X} + (T-\lambda)^{-n}(0)$ is closed in $\mathcal{X}$ [2, Theorem 1.42]. Hence $(T-\lambda)^{-(n+1)}\mathcal{X}$ is closed [31, Lemma 12], which implies that $\lambda \in \mathcal{P}^a(T)$.

Now let $\mathcal{X} = \mathcal{H}$. Then it follows from the above that (v)$\Longrightarrow$ (ii). To prove the reverse implication, we observe that if $\lambda \in \Pi^a(T)$, then $\lambda \notin \sigma_{SBW_+^-}(T)$. There exists a decompositions $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $T-\lambda = T_1 \oplus T_2$ such that $T_1 = (T-\lambda)|_{\mathcal{H}_1}$ is nilpotent and $T_2 = (T-\lambda)|_{\mathcal{H}_2}$ is upper semi–Fredholm [6, Theorem 2.6]. Since semi–Fredholm operators are Kato type, it follows that $T-\lambda$ is Kato type. $\square$

Perturbations. Wt is transmitted from operators $T \in B(\mathcal{X})$ to $T + N$ for commuting nilpotents $N \in B(\mathcal{X})$ [33, Theorem 3]. This, however, fails, for commuting quasinilpotent operators: consider, for example, the operators $T = I \oplus 0 \in B(\mathcal{X} \oplus \mathcal{X})$ and $Q = 0 \oplus S \in B(\mathcal{X} \oplus \mathcal{X})$, where $\mathcal{X} = \ell^2(\mathbf{N})$ and $S : (x_1, x_2, x_3, ...) \to (\frac{1}{2}x_2, \frac{1}{3}x_3, ...)$ on $\ell^2(\mathbf{N})$. Then T satisfies $a - Wt$ (hence, also Wt) but $T + Q$ does not satisfy Wt (hence, also $a - Wt$). Again, if $T = 0 + Q \in B(\mathcal{X})$, where Q is a quasinilpotent, then $\sigma(T) = \sigma_{BW}(T) = \Pi(T) = \{0\}$, and T does not satisfy gWt (hence, also $a - gWt$). Thus the results of Theorem 8.3.10and 8.3.11 do not extend to Weyl's theorem. The situation however is different if we restrict ourselves to injective quasinilpotent operators which commute with operators $T \in B(\mathcal{X})$ with SVEP.

Theorem 8.4.8 *If $T \in B(\mathcal{X})$ has SVEP and commutes with an injective quasi–nilpotent operator $Q \in B(\mathcal{X})$, then $T + Q$ satisfies Wt. Furthermore, if T is finitely isoloid (i.e., isolated points of $\sigma(T)$ are eigenvalues of finite multiplicity), then $T + Q$ satisfies gWt and $T^* + Q^*$ satisfies $a - gWt$.*

Proof. T and $T + Q$ being quasi–nilpotent equivalent, $T + Q$ has SVEP ($\Longrightarrow \sigma(T + Q) = \sigma(T^* + Q^*) = \sigma_a(T^* + Q^*)$). The commutativity of T and Q implies that if $(0 \neq)x \in (T + Q - \lambda)^{-1}(0)$ for some $\lambda \in \sigma(T + Q)$, then $Q^m x \in (T + Q - \lambda)^{-1}(0)$ for all nonnegative integers m. Let $p(t) = \sum_{i=1}^n c_i t^i = c_n \Pi_{i=1}^n (t - \lambda_i)$ be a polynomial such that $p(Q) = 0$. Then the injectivity of Q implies that $c_n = 0$; hence, by a finite induction argument, $c_i = 0$ for all $0 \leq i \leq n$. Since this implies that $\{Q^n x\}$ is a linearly independent set of vectors in $(T + Q - \lambda)^{-1}(0)$, eigenvalues of $T + Q$, hence also of T since $T = (T + Q) - Q$, have infinite multiplicity. In particular, $\Pi_0(T + Q) = \emptyset$. Clearly, $T + Q$ satisfies $a - gBt$; hence $T + Q$ satisfies Bt, i.e., $\sigma(T+Q) \setminus \sigma_W(T+Q) = \mathcal{R}_0(T+Q)$. Since $\mathcal{R}_0(T+Q) \subseteq \Pi_0(T+Q) = \emptyset$, $\sigma(T + Q) = \sigma_w(T + Q)$ and $T + Q$ satisfies Wt.

Assume now that T is finitely isoloid. Then it follows from the above that $\text{iso}\sigma(T) = \text{iso}\sigma(T + Q) = \emptyset$. As earlier noted, $T + Q$ satisfies $a - gBt$; hence $T+Q$ satisfies gBt, i.e., $\sigma(T+Q) \setminus \sigma_{BW}(T+Q) = \mathcal{P}(T+Q) = \emptyset$. Since $\mathcal{P}(T + Q) \subseteq \Pi(T + Q)$, and since $\lambda \in \Pi(T + Q) \Longrightarrow \lambda \in \text{iso}\sigma(T + Q) = \emptyset$, it follows that $\sigma(T+Q) \setminus \sigma_{BW}(T+Q) = \Pi(T+Q)$, i.e., $T+Q$ satisfies gWt. By Lemma 8.3.6, $T^* + Q^*$ satisfies $a - gBt \Longrightarrow \sigma_a(T^* + Q^*) \setminus \sigma_{SBF_+^-}(T^* + Q^*) =$

$\mathcal{P}^a(T^*+Q^*)$ ($\subseteq \Pi^a(T^*+Q^*) = \Pi(T^*+Q^*)$). Since $\lambda \in \Pi(T^*+Q^*) \Longrightarrow \lambda \in \text{iso}\sigma(T+Q) = \emptyset$, it follows that $\sigma_a(T^*+Q^*) \setminus \sigma_{SBF_+^-}(T^*+Q^*) = \Pi^a(T^*+Q^*) = \emptyset$, so that T^*+Q^* satisfies $a-gWt$. (Observe that since $T = (T+Q)-Q$, also T satisfies gWt and T^* satisfies $a-gWt$.) $\square$

$T+Q$ may fail to satisfy gWt, and T^*+Q^* may fail to satisfy $a-gWt$, in the absence of the hypothesis that T is finitely isoloid.

Example 8.4.9 Let $S \in \ell^2(\mathbf{N})$ be the weighted forward unilateral shift with the weight sequence $\{\frac{1}{n+1}\}$. Then S is an injective quasi–nilpotent such that range of Q^n is not closed for every $n \in \mathbf{N}$. Define $Q \in \ell^2(\mathbf{N}) \oplus \ell^2(\mathbf{N})$ by $Q = S \oplus S$. Let $T \in \ell^2(\mathbf{N}) \oplus \ell^2(\mathbf{N})$ be defined by $T = (I-S) \oplus 0$. Then T has SVEP and commutes with Q. Denoting $T+Q$ by A, it is seen that $\sigma(A) = \sigma_w(A) = \sigma_{BW}(A) = \{0,1\}$, $\Pi_0(A) = \emptyset$ and $\Pi(A) = \{1\}$. Evidently, T is not finitely isoloid, A satisfies Wt and A does not satisfy gWt. Again, since $\sigma_a(A^*) = \sigma_{SBF_+^-}(A^*) = \{0,1\}$ and $\Pi^a(A^*) = \{1\}$, A^* does not satisfy $a-gWt$.

Example 8.4.10 This example shows that the commutativity hypothesis in Theorem 8.4.8 is essential. Let S be the injective quasi–nilpotent of Example 8.4.9, and let $T \in \ell^2(\mathbf{N})$ be the nilpotent $Tx = (0, -\frac{1}{2}x_1, 0, \ldots)$. Then T and S do not commute, $\sigma(T+S) = \sigma_w(T+S) = \Pi_0(T+S) = \{0\}$, and $T+S$ does not satisfy Wt.

We say that an operator $T \in B(\mathcal{X})$ is *polynomially polaroid* if there exists a non–trivial polynomial $g(.)$ such that $g(T)$ is polaroid (i.e., all isolated points of $\sigma(g(T))$ are poles of the resolvent of $g(T)$). Polynomially polaroid operators are polaroid (see Example 5.5 below). We shall require the following technical lemma in the proof of our next theorem.

Lemma 8.4.11 *If $T \in B(\mathcal{X})$ is polaroid, and if $TN = NT$ for some nilpotent operator $N \in B(\mathcal{X})$, then there exists an $m \in \mathbf{N}$ such that*

$$H_0(T+N-\lambda) = (T+N-\lambda)^{-m}(0),$$

at points $\lambda \in iso\sigma(T)$.

Proof. We may assume that $N^t = 0$ for some $t \in \mathbf{N}$. Choose an integer $n > t$. Then, for every $x \in \mathcal{X}$ and complex number λ,

$$\begin{aligned} ||(T-\lambda)^n x||^{\frac{1}{n}} &= ||((T+N-\lambda)-N)^n x||^{\frac{1}{n}} \\ &= ||\sum_{j=0}^{m-1} (-1)^j \ {}^nC_j N^j (T+N-\lambda)^{n-j} x||^{\frac{1}{n}} \\ &\leq \sum_{j=0}^{m-1} [{}^nC_j ||N||^j]^{\frac{1}{n}} ||(T+N-\lambda)^{n-j} x||^{\frac{1}{n}}, \end{aligned}$$

which implies that

$$H_0(T-\lambda) \subseteq H_0(T+N-\lambda).$$

By symmetry,

$$H_0(T+N-\lambda) \subseteq H_0(T+N-\lambda-N) = H_0(T-\lambda).$$

Hence

$$H_0(T-\lambda) = H_0(T+N-\lambda).$$

Choose $\lambda \in \text{iso}\sigma(T)$. Then $H_0(T-\lambda) = (T-\lambda)^{-p}(0)$ for some $p \in \mathbf{N}$. Set $m = pt$. Since

$$\begin{aligned} & x \in (T-\lambda)^{-m}(0) \Longrightarrow (T+N-\lambda)^m x \\ = \ & (T-\lambda)^p \{(T-\lambda)^{m-p} +^m C_1 (T-\lambda)^{m-p-1} N + ... +^m C_{m-p} N^{m-p}\} x \\ + \ & \{{}^m C_{m-p+1} (T-\lambda)^{p-1} N^{m-p-t+1} + ... + N^{m-y}\} N^t x = 0, \end{aligned}$$

it follows that

$$H_0(T+N-\lambda) = (T-\lambda)^{-m}(0) \subseteq (T+N-\lambda)^{-m}(0).$$

Finally, since $(T+N-\lambda)^{-m}(0) \subseteq H_0(T+N-\lambda)$ for all integers $m \geq 1$, the conclusion follows. $\square$

Theorem 8.4.12 *Let $T \in B(\mathcal{X})$ be a polynomially polaroid operator with SVEP, and let $A \in B(\mathcal{X})$ be an algebraic operator which commutes with T. Then $f(T+A)$ satisfies gWt for every $f \in H(\sigma(T+A))$ and $f(T^*+A^*)$ satisfies $a-gWt$ for every $f \in H_c(\sigma(T+A))$.*

Proof. Suppose that $g(T)$ is polaroid and has SVEP for some polynomial g. Then T has SVEP. Defining μ_i, A_i and T_i, $1 \leq i \leq n$, just as in the proof of Theorem 8.3.13, it is seen that $T_i + A_i$ (for all $1 \leq i \leq n$) and $T + A$ have SVEP. Recall that the hypothesis $g(T)$ is polaroid for some polynomial $g(.)$ implies that T is polaroid; hence Lemma 8.4.11 implies that $H_0(T_i + A_i - \mu_i - \lambda) = (T_i + A_i - \mu_i - \lambda)^{-p}(0)$ for some $p \in \mathbf{N}$ at points $\lambda \in \text{iso}\sigma(T)$. We prove next that $H_0(T + A - \lambda) = (T + A - \lambda)^{-m}(0)$, for some $m \in \mathbf{N}$, at every $\lambda \in \text{iso}\sigma(T + A)$.

Clearly, for every $\lambda \in \text{iso}\sigma(T + A)$, either $\lambda - \mu_i \notin \sigma(T_i)$ or $\lambda - \mu_i \in \text{iso}\sigma(T_i)$; $1 \leq i \leq n$. If $\lambda - \mu_i \notin \sigma(T_i)$, then $T_i - (\lambda - \mu_i)$ is invertible, which implies that $\{T_i - (\lambda - \mu_i)\} + \{A_i - \mu_i\}$ is invertible, and hence that

$$H_0(T_i + A_i - \lambda) = H_0((T_i + A_i - \mu_i) - (\lambda - \mu_i)) = \{0\} = (T_i + A_i - \lambda)^{-m_i}(0).$$

If, instead, $\lambda - \mu_i \in \text{iso}\sigma(T_i)$, then, by Lemma 8.4.11,

$$\begin{aligned} H_0(T_i + A_i - \lambda) &= H_0((T_i + A_i - \mu_i) - (\lambda - \mu_i)) \\ &= ((T_i + A_i - \mu_i) - (\lambda - \mu_i))^{-m_i}(0) \\ &= (T_i + A_i - \lambda)^{-m_i}(0), \end{aligned}$$

for some $m_i \in \mathbf{N}$. Let $m = \max\{m_1, m_2, ..., m_n\}$. Then

$$\begin{aligned} H_0(T + A - \lambda) &= \bigoplus_{i=1}^{n} H_0(T_i + A_i - \lambda) = \bigoplus_{i=1}^{n} (T_i + A_i - \lambda)^{-m_i}(0) \\ &= (T + A - \lambda)^{-m}(0). \end{aligned}$$

The conclusion that $H_0(T+A-\lambda) = (T+A-\lambda)^{-m}(0)$, for some $m \in \mathbf{N}$, at every $\lambda \in \text{iso}\sigma(T + A)$ implies that $T + A$ is polaroid. Since $T + A$ has SVEP, Proposition 8.4.4 applies. $\square$

8.5. Examples

As seen above, the implications $gWt \Longrightarrow Wt$ and $a - gWT \Longrightarrow a - Wt$ can not in general be reversed. There is however a class of operators, namely the class of "hereditarily polaroid operators" for which the implication $gWt \Longrightarrow Wt$ is an equivalence; additionally, for an operator T in this class of operators, T^* satisfies Wt is equivalent to T^* satisfies $a-gWt$. A part of an operator is its restriction to an invariant subspace. We say that an operator $T \in B(\mathcal{X})$ is *hereditarily polaroid*, $T \in \mathcal{HP}$, if every part of T is polaroid. It is easily seen that the $\mathcal{HP}$ property is similarity invariant. However, $\mathcal{HP}$ property is not preserved by quasi–affinities. Thus, let $T \in \ell^2(\mathbf{N})$ be the forward unilateral shift, let $S \in \ell^2(\mathbf{N})$ be the weighted forward unilateral shift with the weight sequence $\{\frac{1}{n+1}\}$ and let $A \in \ell^2(\mathbf{N})$ be the multiplication operator defined by $Ax = \{\frac{x_n}{n!}\}_{n\in\mathbf{N}}$ for all $x = \{x_n\}_{n\in\mathbf{N}} \in \ell^2(\mathbf{N})$. Then A is injective, has dense range and satisfies $SA = AT$. Evidently, $T \in \mathcal{HP}$ and S is quasi–nilpotent. A uniformly convergent sequence of $\mathcal{HP}$ operators may not converge to an $\mathcal{HP}$ operator: consider a sequence of nilpotent operators which converges in the norm topology to a quasi–nilpotent operator. The sum of an $\mathcal{HP}$ operator with a quasi–nilpotent operator is (generally) not an $\mathcal{HP}$ operator: consider $T = 0 + Q$, where Q is a quasi–nilpotent.

The class of $\mathcal{HP}$ operators is large. The following examples show that it includes a number of the more commonly considered classes of operators.

Example 8.5.1 If $H(p)$ denotes the class of operators $T \in B(\mathcal{X})$ such that for every $\lambda \in \mathbf{C}$ $H_0(T-\lambda) = (T-\lambda)^{-p_\lambda}(0)$ for some non–negative integer p_λ, then $H(p) \subset \mathcal{HP}$. Evidently, if M is a T-invariant subspace, and $H_0(T-\lambda) = (T-\lambda)^{-p_\lambda}(0)$, then $H_0((T-\lambda)|_M) \subseteq (T-\lambda)^{-p_\lambda}(0) \cap M = ((T-\lambda)|_M)^{-p_\lambda}(0)$. Since $((T-\lambda)|_M)^{-p_\lambda}(0) \subseteq H_0((T-\lambda)|_M)$ for every $p_\lambda \in \mathbf{N}$, $H_0((T-\lambda)|_M) = (T-\lambda)|_M{}^{-p_\lambda}(0)$. Now let $\lambda \in \text{iso}\sigma((T-\lambda)|_M)$. Then

$$\begin{aligned}
& M = H_0((T-\lambda)|_M) \oplus K((T-\lambda)|_M) \\
& \quad = (T-\lambda)|_M{}^{-p_\lambda}(0) \oplus K((T-\lambda)|_M) \\
\Longrightarrow \quad & (T-\lambda)|_M{}^{p_\lambda} M = 0 \oplus (T-\lambda)|_M{}^{p_\lambda} K((T-\lambda)|_M) = K((T-\lambda)|_M) \\
\Longrightarrow \quad & M = (T-\lambda)|_M{}^{-p_\lambda}(0) \oplus (T-\lambda)|_M{}^{p_\lambda} M,
\end{aligned}$$

i.e., λ is a pole of the resolvent of $(T-\lambda)|_M$. Finally, since there exist $\mathcal{HP}$ operators which are not $H(p)$ operators, for example “quasihyponormal operators ”[3], $H(p) \subset \mathcal{HP}$.

The class $H(p)$ is large [2. p. 170-176]. It contains, amongst other classes, the classes consisting of p-hyponormal operators $(T \in B(\mathcal{H}) : |T^*|^{2p}$

$\leq |T|^{2p}$ for some $0 < p \leq 1$), M-hyponormal operators ($T \in B(\mathcal{H}) : ||(T - \lambda)^*x||^2 \leq M||(T-\lambda)x||^2$ for some $M \geq 1$, all $\lambda \in \mathbf{C}$ and $x \in \mathcal{H}$), totally $*$-paranormal operators ($T \in B(\mathcal{H}) : ||(T-\lambda)^*x||^2 \leq ||(T-\lambda)^2x||$ for every unit vector $x \in \mathcal{H}$ and all $\lambda \in \mathbf{C}$), totally paranormal operators ($T \in B(\mathcal{X}) : ||(T-\lambda)x||^2 \leq ||(T-\lambda)^2x||$ for every unit vector $x \in \mathcal{H}$ and all $\lambda \in \mathbf{C}$), transaloid operators ($T \in B(\mathcal{X}) : ||T-\lambda||$ equals the spectral radius $r(T-\lambda)$ for all $\lambda \in \mathbf{C}$), generalized scalar and subscalar operators, and multipliers of commutative semi-simple Banach algebras. (Here, as before, $\mathcal{H}$ denotes a Hilbert space.)

Example 8.5.2 $T \in B(\mathcal{X})$ is *totally hereditarily normaloid*, $T \in \mathcal{THN}$, if every part, and T_p^{-1} for every invertible part T_p, of T is normaloid (i.e., the norm of the part equals its spectral radius); T is *completely hereditarily normaloid*, $T \in \mathcal{CHN}$, if either $T \in \mathcal{THN}$ or $T - \lambda$ is normaloid for every $\lambda \in \mathbf{C}$. $\mathcal{CHN}$ operators are *simply hereditarily polaroid*, i.e., the poles of every part of the operator are simple (or order one) [12, Proposition 2.1]. In particular, paranormal operator (i.e., operators $T \in B(\mathcal{X})$ such that $||Tx||^2 \leq ||T^2x||$ for every unit vector $x \in \mathcal{X}$ [26, p 229]) are simply $\mathcal{HP}$ operators.

Example 8.5.3 A Hilbert space operator $T \in B(\mathcal{H})$ is (p,k)-quasihyponormal, $T \in (p,k) - Q$, for some integer $k \in \mathbf{N}$ and $0 < p \leq 1$, if $T^{*k}(|T|^{2p} - |T^*|^{2p})T^k \geq 0$ [27]. The restriction of a $(p,k) - Q$ operator to an invariant subspace is again $(p,k) - Q$ [27]. Since $(p,k) - Q$ operators are polaroid [13], $(p,k) - Q$ operators are $\mathcal{HP}$.

Example 8.5.4 $T \in B(\mathcal{H})$ is a 2-isometry (or, a 2-isometric operator) if $T^{*2}T^2 - 2T^*T + I = 0$. Every 2-isometric operator is left invertible; if T is not invertible then $\sigma(T)$ is the closed unit disc ($\text{iso}\sigma(T) = \emptyset$), and if T is invertible then it is a unitary [1]. Evidently, the restriction of a 2-isometry to an invariant subspace is a 2-isometry. Hence, 2-isometric operators are $\mathcal{HP}$.

Example 8.5.5 An operator $T \in B(\mathcal{X})$ is *polynomially* $\mathcal{HP}$ if there exists a non–trivial polynomial g such that $g(T) \in \mathcal{HP}$. Polynomially $\mathcal{HP}$ operators are $\mathcal{HP}$, as the following argument shows. Let $A = T|_M$, where M is an invariant subspace of T; let $A_0 = A|_{H_0(A-\lambda)}$ and $A_1 = A|_{K(A-\lambda)}$. If $\lambda \in \mathrm{iso}\sigma(A)$, then $M = H_0(A-\lambda) \oplus K(A-\lambda)$, $\sigma(A_0) = \{\lambda\}$ and A_1 is invertible. Evidently, $\sigma(g(A_0)) = \{g(\lambda)\}$ and (since $g(A)$ is polaroid) there exists a positive integer n such that $H_0(g(A) - g(\lambda)) = (g(A) - g(\lambda))^{-n}(0)$ ($\Longleftrightarrow (g(A)-g(\lambda))^n = 0$). Letting $(g(A_0)-g(\lambda))^n = 0 = c_0(A_0-\lambda)^t \Pi_{i=1}^s (A_0 - \lambda_i)$ for some scalars c_0 and λ_i ($1 \leq i \leq s$), and positive integers s and t, it follows that $(A_0 - \lambda)^t = 0 \Longrightarrow H_0(A_0 - \lambda) = (A - \lambda)^{-t}(0)$. Hence $M = (A-\lambda)^{-t}(0) \oplus K(A-\lambda) \Longrightarrow M = (A-\lambda)^{-t}(0) \oplus (A-\lambda)^t M$, i.e., λ is a pole of the resolvent of A.

Example 8.5.6 Let $m \in \mathbf{N}$. We say that $T \in B(\mathcal{X})$ satisfies a local growth condition of order m, $T \in \mathrm{loc}(G_m)$, if for every closed set $F \subset \mathbf{C}$ and every $x \in X_T(F)$ there exists an analytic function $f : \mathbf{C} \setminus F \longrightarrow \mathcal{X}$ such that $(T - \lambda)f(\lambda) \equiv x$ and $||f(\lambda)|| \leq K[\mathrm{dist}(\lambda, F)]^{-m}||x||$ for some $K > 0$ (independent of F and x). Hyponormal operators are $\mathrm{loc}(G_1)$ [28] and spectral operators of type $m-1$ are $\mathrm{loc}(G_m)$ [20, Proof of Theorem XV.6.7]. Evidently, $T \in \mathrm{loc}(G_m)$ satisfy a growth condition of order m (so that $T \in \mathrm{loc}(G_m) \Longrightarrow T \in (G_m)$); since (G_m) operators are polaroid, see for example [10, Lemma 3], $\mathrm{loc}(G_m)$ operators are polaroid. However, $\mathrm{loc}(G_m)$ operators are not $\mathcal{HP}$. To see this, let $T = U \oplus Q \in B(\mathcal{H} \oplus \mathcal{H})$, where (as before) U is the forward unilateral shift and Q is a (non–nilpotent) quasi–nilpotent operator. Then $T \in \mathrm{loc}(G_1)$ is polaroid ($\mathrm{iso}\sigma(T) = \emptyset$); however, $T \notin \mathcal{HP}$ since Q is not polaroid.

Not every polaroid operator has SVEP: consider for example the backward unilateral shift, which is trivially polaroid (since its spectrum has no isolated points) and which does not have SVEP. In contrast,
$\mathcal{HP}$ operators have SVEP [15, Theorem 2.8].
Furthermore, $\Pi(T) = \mathcal{P}(T) = \mathcal{P}(T^*) = \Pi(T^*)$ *for operators* $T \in \mathcal{HP}$. The following theorem is immediate from Theorem 8.4.12, Example 8.5.6, and the following observation: if $g(T) \in \mathrm{loc}(G_m)$ for some nontrivial polynomial $g(.)$ and $\mathcal{X}$ is reflexive, then $g(T)$, hence also T, has SVEP [28].

Theorem 8.5.7 *(i) If $T \in B(\mathcal{X})$ is a polynomially $\mathcal{HP}$ operator which commutes with the algebraic operator $A \in B(\mathcal{X})$, then $f(T + A)$ satisfies gWt for every $f \in H(\sigma(T + A))$ and $f(T^* + A^*)$ satisfies $a - gWt$ for every*

$f \in H_c(\sigma(T+A))$.
(b) If the Banach space $\mathcal{X}$ is reflexive (in particular, a Hilbert space) and T is a polynomially $loc(G_m)$ operator which commutes with the algebraic operator $A \in B(\mathcal{X})$, then $f(T+A)$ satisfies gWt for every $f \in H(\sigma(T+A))$ and $f(T^*+A^*)$ satisfies $a-gWt$ for every $f \in H_c(\sigma(T+A))$.

Hypercyclic and supercyclic operators. For a separable Banach space $\mathcal{X}$, an operator $T \in B(\mathcal{X})$ is *hypercyclic*, $T \in \mathcal{HC}$, if there exists a vector $x \in \mathcal{X}$ such that the orbit $\mathrm{orb}(T;x) = (x, Tx, T^2x, ...)$ is dense in $\mathcal{X}$; T is *supercyclic*, $T \in \mathcal{HS}$, if there exists a vector $x \in \mathcal{X}$ such that the set of scalar multiples of $\mathrm{orb}(T;x)$ is dense in $\mathcal{X}$. Evidently, $\mathcal{HC} \subset \mathcal{SC}$. Recall ([21], [24]) that if $T \in \mathcal{HC}$ (resp., $T \in \mathcal{SC}$), then $\sigma_p(T^*) = \emptyset$ (resp., either $\sigma_p(T^*) = \emptyset$ or $T = S \oplus \alpha I$, where $0 \neq \alpha \notin \sigma_b(T)$ and $\frac{1}{\alpha}S \in \mathcal{HC}$. If we let $\mathcal{C} = \mathcal{HC} \cup \mathcal{SC}$, then T^* has SVEP for operators $T \in \mathcal{C}$. Hence:
(I). *If $T \in \mathcal{C}$, then $f(T)$ and $f(T^*)$ satisfy $a-gBt$ for every $f \in H(\sigma(T))$.*
(II). *If $R \in B(\mathcal{X})$ is a Riesz operator which commutes with $T \in \mathcal{C}$, then $f(T+R)$ and $f(T^*+R^*)$ satisfy $a-gBt$ for every $f \in H(\sigma(T))$. Furthermore, if R is an injective quasinilpotent, then $f(T+R)$ satisfies $a-gWt$ for every $f \in H_c(\sigma(T))$.*
(III). *If $A \in B(\mathcal{X})$ is an algebraic operator which commutes with $T \in \mathcal{C}$, then $f(T+A)$ and $f(T^*+A^*)$ satisfy $a-gBt$ for every $f \in H_c(\sigma(T))$.*
More is true.

Theorem 8.5.8 *If $T \in \mathcal{C}$, then T^* satisfies gWt. If also $\Pi(T) \subseteq \Pi(T^*)$, then T satisfies $a-gWt$.*

Proof. If $T \in \mathcal{HC}$, or $T \in \mathcal{SP}$ and $\sigma_p(T^*) = \emptyset$, then $\mathcal{P}(T) = \mathcal{P}(T^*) = \Pi(T^*) = \emptyset$. Hence, since T^* satisfies gBt, $\sigma(T^*) \setminus \sigma_{BW}(T^*) = \mathcal{P}(T^*) \subseteq \Pi(T^*) = \emptyset$, which implies that T^* satisfies gWt. Now let $T \in \mathcal{SC}$ be such that $\sigma_p(T^*) = \{\alpha\}$ for some nonzero $\alpha \notin \sigma_b(T^*)$ $(= \sigma_b(T))$. Then $\alpha \in \mathrm{iso}\sigma(T^*)$ $(= \mathrm{iso}\sigma(T))$ is a pole of the resolvent of T^*. Since T^* satisfies gBt, $\sigma(T^*) \setminus \sigma_{BW}(T^*) = \mathcal{P}(T^*) = \Pi(T^*) = \{\alpha\}$. Once again, T^* satisfies gWt. Assume now that $\Pi(T) \subseteq \Pi(T^*)$. Since T^* has SVEP implies $\sigma(T^*) = \sigma(T) = \sigma_a(T)$, $\Pi(T) = \Pi^a(T)$ and $\sigma_{SBW_+^-}(T) = \sigma_{BW}(T) = \sigma_{BW}(T^*)$, and (since T^* satisfies $a-gBt \Longrightarrow$) $\sigma_a(T) \setminus \sigma_{SBW_+^-}(T) = \sigma(T^*) \setminus \sigma_{BW}(T^*) =$

$\Pi(T^*)$. Again, since T satisfies gBt, $\sigma(T) \setminus \sigma_{BW}(T) = \mathcal{P}(T) \subseteq \Pi(T) \subseteq \Pi(T^*)$; hence $\sigma_a(T) \setminus \sigma_{SBW_+^-}(T) = \Pi^a(T)$, i.e., T satisfies $a - gWt$. □

The hypothesis $\Pi(T) \subseteq \Pi(T^*)$ is essential in Theorem 8.5.8 for T to satisfy $a-gWt$. To see this consider the backward unilateral shift $T \in B(\mathcal{H})$. Then T is supercyclic, T^* satisfies gWt and T^* does not satisfy $a - gWt$ (even, Wt). Evidently, $\Pi(T) \nsubseteq \Pi(T^*)$.

It is evident from the above that if $T \in \mathcal{C}$ is such that $\sigma_a(T)$ is connected, then T, T^* satisfy Weyl's theorem and T satisfies a-Weyl's theorem. A sufficient condition for an operator T to have connected spectrum is that the *hyper-range* $\cap_{n=1}^{\infty} T^n\mathcal{X} = \{0\}$ [25, Proposition 2]; alternatively, if T is surjective and the closure of the *hyper-kernel* $\overline{\mathcal{N}(T)} = \overline{\cup_{n=1}^{\infty} T^{n-1}(0)} = \mathcal{X}$, then $(\mathcal{X} = \overline{\mathcal{N}(T)} =^{\perp} (\cap_{n=1}^{\infty} T^{*n}\mathcal{X}^*) =^{\perp} \{0\} \Longrightarrow \sigma(T^*) =) \ \sigma(T)$ is connected.

Let

$$\rho_k(T) = \{\lambda \in \mathbf{C} : (T-\lambda)^{-1}(0) \subset \cap_{n=1}^{\infty}(T-\lambda)^n\mathcal{X}\}$$

denote the Kato resolvent of T.

Corollary 8.5.9 *If $\overline{\mathcal{N}(T-\lambda)} = \mathcal{X}$ for some $\lambda \in \rho_k(T)$, and if f is a non-constant analytic function on a neighbourhood of $\sigma(T)$, then $f(T)$ satisfies a-Weyl's theorem and $f(T^*)$ satisfies Weyl's theorem.*

Proof. $f(T) \in \mathcal{SC}$ [25, Theorem 3]. If C_λ denotes the component of $\rho_k(T)$ such that $\lambda \in C_\lambda$, then the invariance of the hyper-kernel $\mathcal{N}(T-\lambda)$ implies that $\overline{\mathcal{N}(T-\mu)} = \mathcal{X}$ for every $\mu \in C_\lambda$. In particular, $T-\mu$ is surjective for every $\mu \in C_\lambda$, which implies that $\sigma(T-\mu)$ is connected. Hence $\sigma(T)$ is connected. □

Let $T \in \mathcal{HC}$. Then T^* has SVEP and it follows that $\sigma_e(T) = \sigma_{SF_+}(T) = \{\lambda \in \mathbf{C} : T-\lambda \notin \Phi_{SF_+}(\mathcal{X})\}$. Since T^* has no eigenvalues, $\lambda \in \sigma_a(T^*)$ implies that either $\dim H_0(T^*-\lambda I^*) = \infty$ or $\alpha(T^*-\lambda I^*) = 0$. Trivially, $\dim H_0(T^*-\lambda I^*) = \infty$ implies that $\lambda \in \sigma_{SF_+}(T^*)$; since $\alpha(T^*-\lambda I^*) = 0$ implies that $T^*-\lambda I^*$ is not bounded below if and only if $(T^*-\lambda I^*)\mathcal{X}$ is not closed, again $\lambda \in \sigma_{SF_+}(T^*)$. Hence $\sigma_a(T^*) \subseteq \sigma_{SF_+}(T^*)$. Since the reverse inclusion holds for every operator, $\sigma_a(T^*) = \sigma_{SF_+}(T^*)$.

Let $\partial\mathbf{D}$ denote the boundary of the unit disc $\mathbf{D}$ in $\mathbf{C}$, and let $\sigma_{SF_-}(T) = \{\lambda \in \mathbf{C} : T - \lambda \notin \Phi_{SF_-}(\mathcal{X})\}$.

Corollary 8.5.10 *If $T \in \mathcal{HC}$ is such that $\sigma_x(T) \cap \partial\mathbf{D} = \emptyset$, where $\sigma_x(T) = \sigma_e(T)$ or $\sigma_{SF_+}(T)$ or $\sigma_{SF_-}(T)$ or $\partial\sigma_e(T)$, then T satisfies $a - gWt$ and T^* satisfies gWt.*

Proof. We prove that $\sigma(T)$ is connected. We have already seen that $\sigma_a(T^*) = \sigma_{SF_+}(T^*)$. Since $\partial\sigma(T^*) \subseteq \sigma_a(T^*)$ and $\sigma_{SF_+}(T^*) \subseteq \sigma_e(T^*)$, it follows that $\partial\sigma(T^*) \subseteq \partial\sigma_e(T^*) \Longrightarrow \partial\sigma(T) \subseteq \partial\sigma_e(T)$. Recall that $\partial\sigma_e(T) \subseteq \sigma_{SF_+}(T) \cap \sigma_{SF_-}(T)$ for every operator T.

Hence

$$\partial\sigma(T) \subseteq \partial\sigma_e(T) \subseteq \sigma_{SF_-}(T) \subseteq \sigma_e(T),$$

which implies that if $\sigma_x(T) \cap \partial\mathbf{D} = \emptyset$, then $\partial\sigma(T) \cap \partial\mathbf{D} = \emptyset$. Assume (to the contrary) that $\sigma(T) = \sigma_1 \cup \sigma_2$ for some complementary spectral sets σ_1 and σ_2. Then $\partial\mathbf{D}$ being a connected compact set either $\partial\mathbf{D} \subseteq \sigma_1$ or $\partial\mathbf{D} \subseteq \sigma_2$. Hence either $\partial\mathbf{D} \cap \sigma_1 = \emptyset$ or $\partial\mathbf{D} \cap \sigma_2 = \emptyset$. Since this contradicts the fact that every component of the spectrum of a hypercyclic operator has a non-void intersection with $\partial\mathbf{D}$ [29], we conclude that $\sigma(T)$ is connected. $\square$

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9

Approximation in Hölder norm by linear operators

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Abstract: We present some results concerning approximation in Hölder norms by linear operators. In particular we analyze Cèsaro operators.

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9.1 Introduction

In the last years some researchers have paid attention to the approximation theory in the so called Hölder spaces. This work is developed within this framework.

Important results concerning the best approximation in Hölder norms by trigonometric and algebraic polynomials in L_p spaces were established in [1] and [2] respectively. But there are some interesting approximation processes which were not discussed. In this paper we fill part of this gap by extending the theory to two kinds of approximation methods: the first includes singular integrals (Section 3) and the second one summation processes, as the Riesz and Cesàro means (Section 4). The rate of convergence of the processes are given in terms of certain moduli of smoothness, except for the Riesz means where the estimate is given using a K-functional. It is an open problem to construct a modulus of smoothness equivalent the K-functional presented here as a characterization of the rate of convergence of the Riesz means (see Theorem 9.4.2).

In [1] it was presented a general approach to obtain direct and converse results in Hölder spaces. In particular, if we are able to prove that the operators have a *shape preserving property*, then some good direct estimations can be derived. In Section 2 we present a slight generalization of some of the arguments presented in [1]. This allows us to simplify the presentation of the new results.

9.2 Abstract Hölder spaces

For convenience of the reader we present here some of the results given in [1]. Since it will be used for different spaces it is convenient to work with Banach spaces. In this section E is a Banach space.

Denote by $\Lambda[0,\infty)$ the family of all increasing functions $\lambda : [0,\infty) \to [0,\infty)$ such that $\lambda(0) = 0$ and $\lambda(s) > 0$ for $s > 0$.

Fix a function $\omega : E \times [0,+\infty) \to \mathbb{R}^+$ such that: *a*) For each fixed $t \in (0,+\infty)$, the function $\omega(\cdot,t)$ is a seminorm on E and for all $f \in E$, $\omega(f,0) = 0$; *b*) For each fixed $f \in E$, the function $\omega(f,\cdot)$ is increasing on

$[0, +\infty)$ and continuous at 0; c) There exists a constant $C > 0$ such that for each $(f, t) \in E \times [0, +\infty)$, one has

$$\omega(f, t) \leq C \|f\|. \tag{9.1}$$

Given a real $r > 0$ denote $Ker(\omega) = \{g \in E : \omega(g, t) = 0, t \geq 0\}$ and, for $\lambda \in \Lambda[0, \infty)$,

$$N(E, \omega, \lambda) = \left\{ f \in E : \quad \sup_{t>0} \frac{\omega(f, t)}{\lambda(t)} < \infty \right\}.$$

Assume that $N(E, \omega, \lambda) \neq Ker(\omega)$.

For $f \in N(E, \omega, \lambda)$ we denote $\theta_{\omega,\lambda}(f, 0) = 0$,

$$\theta_{\omega,\lambda}(f, t) = \sup_{0<s\leq t} \frac{\omega(f, s)}{\lambda(s)}, \quad \theta_{\omega,\lambda}(f) = \sup_{s>0} \frac{\omega(f, s)}{\lambda(s)} \tag{9.2}$$

and

$$\|f\|_{\omega,\lambda} = \|f\|_E + \theta_{\omega,\lambda}(f). \tag{9.3}$$

The generalized Hölder space $E_{\omega,\lambda}$ is formed by those $f \in E$ such that $\|f\|_{\omega,\lambda} < \infty$ with the norm $\|f\|_{\omega,\lambda}$. Moreover we denote

$$E^0_{\omega,\lambda} = \left\{ f \in E_{\omega,\lambda} : \quad \lim_{t\to 0} \theta_{\omega,\lambda}(f, t) = 0 \quad \right\}. \tag{9.4}$$

The following theorem is similar to a result presented in [1], but it is more general and is stated a simpler form. After the proof we explain how it works in concrete situations.

Theorem 9.2.1 *Let λ, E and ω be given as above. For each $\rho \in [1, \infty)$ let $M_\rho : E \to E$ be a bounded linear operator such that the following conditions hold: there exist positive constants D_1 and D_2 such that for each $f \in E$, every $\rho \geq 1$ and $t > 0$*

$$\|f - M_\rho f\|_E \leq D_1 \omega\left(f, \psi(1/\rho)\right) \quad \textit{and} \quad \omega(M_\rho f, t) \leq D_2 \omega(f, t), \tag{9.5}$$

where $\psi : (0, 1] \to (0, \infty)$ is a decreasing function and $\psi(s) \to 0$ as $s \to 0$.

Then there exists a constant C such that, for $f \in E^0_{\omega,\lambda}$, one has

$$\|f - M_\rho f\|_{\omega,\lambda} \leq C \theta_{\omega,\lambda}\left(f, \psi\left(\frac{1}{\rho}\right)\right). \tag{9.6}$$

Proof. Fix $f \in E^0_{\omega,\lambda}$ and $\rho \geq 1$. Taking into account the first inequality in (9.5), we have

$$\|f - M_\rho f\|_E \leq D_1 \omega(f, \psi(1/\rho))$$
$$\leq \lambda(\psi(1/\rho))\theta_{\omega,\lambda}\left(f, \psi\left(\frac{1}{\rho}\right)\right) \leq \lambda(\psi(1))\theta_{\omega,\lambda}\left(f, \psi\left(\frac{1}{\rho}\right)\right).$$

In order to estimate the terms $\theta_{\omega,\lambda}(f - M_\rho f)$ we consider two cases.

Case 1. If $0 < s < \psi(1/\rho)$, since $\omega(\cdot, s)$ is seminorm on E we use the second inequality in (9.5) to obtain

$$\omega(f - M_\rho f, s) \leq (1 + D_2)\omega(f, s) = (1 + D_2)\frac{\omega(f, s)}{\lambda(s)}\lambda(s)$$
$$\leq (1 + D_2)\theta_{\omega,\lambda}(f, s)\lambda(s) \leq (1 + D_2)\theta_{\omega,\lambda}(f, \psi(1/\rho))\lambda(s),$$

since $\theta_{\omega,\lambda}(f, \cdot)$ is increasing.

Case 2. Now assume that $\psi(1/\rho) \leq s$. In this case we consider (9.1) and the first inequality in (9.5) to obtain

$$\omega(f - M_\rho f, s) \leq C\|f - M_\rho f\|_E \leq CD_1\omega\left(f, \psi\left(\frac{1}{\rho}\right)\right)$$
$$\leq CD_1\theta_{\omega,\lambda}\left(f, \psi\left(\frac{1}{\rho}\right)\right)\lambda\left(\psi\left(\frac{1}{\rho}\right)\right) \leq CD_1\theta_{\omega,\lambda}\left(f, \psi\left(\frac{1}{\rho}\right)\right)\lambda(s),$$

since λ is increasing.

We have proved that

$$\theta_{\omega,\lambda}(f - M_\rho f) \leq \max\{1 + D_2, CD_1\}\,\theta_{\omega,\lambda}\left(f, \psi\left(\frac{1}{\rho}\right)\right). \square$$

In applications E is a Banach space of continuous or integrable functions and ω is a modulus of smoothness of a certain order r. The classical Hölder spaces are obtained by taking $\lambda(s) = s^\alpha$ with $0 < \alpha < r$. If $\{M_\rho\}$ is an approximation process on E, then by the Banach-Steinhauss theorem the norms of the operators is uniformly bounded. The first inequality in (9.5) is the typical direct result in terms of the modulus of smoothness. Thus, theorem above says that, if the approximation process has the global smoothness preservation property (second inequality in (9.5)), then a similar estimate holds for the approximation with the same family, in the generalized Hölder space.

9.3 Singular integrals

In this section we denote by X the space $C_{2\pi}$ of 2π-periodic continuous function or $L^p_{2\pi}$, $p \geq 1$, the corresponding Lebesgue space.

Let $\{\chi_\rho\}_{\rho \in I}$ be a family of functions such that, for each $\rho \in I$, $\chi_\rho \in L^1_{2\pi}$, and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_\rho(u) du = 1. \tag{9.7}$$

We associate to $\{\chi_\rho\}$ the singular integral

$$I_\rho(f, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u)\chi_\rho(u) du, \quad f \in X. \tag{9.8}$$

Notice that

$$\|I_\rho(f)\|_X \leq \|\chi_\rho\|_{L^1_{2\pi}} \|f\|_X. \tag{9.9}$$

For a function $f \in X$ the second order modulus of smoothness is defined by

$$\omega_2(f, h)_X = \sup_{t \in (0,h]} \|\Delta_h^2 f\|_X,$$

where $\Delta_h^2 f(,x) = f(x+2h) - 2f(x+h) + f(x)$.

Theorem 9.3.1 *Let $\chi_\rho \in L^1_{2\pi}$ be an even positive function for which* (9.7) *holds. If I_ρ is given by* (9.8) *then , for any $f \in X$,*

$$\|f - I_\rho(f)\|_X \leq \left(\frac{1}{\sqrt{2}} + \frac{\pi}{2} \right) \omega_2 \left(f, \sqrt{1 - \widehat{\chi}_\rho(1)} \right)_X, \tag{9.10}$$

where $\widehat{\chi}_\rho(1)$ is the first Fourier coefficient of χ_ρ. Moreover for $h > 0$,

$$\omega_2 \left(I_\rho(f), h \right)_X \leq \omega_2(f, h)_X.$$

Proof. The inequality (9.10) is well known (see [3], p. 70). For the second inequality notice that

$$\Delta_s^2 I_\rho(f, x) = I_\rho(\Delta_s^2 f, x).$$

Thus, it follows from (9.9) and (9.7) that

$$\|\Delta_s^2 I_\rho(f)\|_X \leq \|\chi_\rho\|_{L^1_{2\pi}} \|\Delta_s^2 f\|_X \leq \|\chi_\rho\|_{L^1_{2\pi}} \omega_2(f, s)_X = \omega_2(f, s)_X. \square$$

The next result follows from Theorems 9.2.1 and 9.3.1. We take $E = X$ and $w(f,\cdot) = \omega_2(f,\cdot)$ in (9.4). Moreover we write $E^0_{2,\lambda}$ and $\theta_{2,\lambda}$ in place of $E^0_{\omega,\lambda}$ and $\theta_{\omega,\lambda}$.

Theorem 9.3.2 *Let $\{\chi_\rho\}_{\rho\in I}$, $\chi_\rho \in L^1_{2\pi}$, be a family of even positive functions for which* (9.7) *holds and*

$$\lim_{\rho\to\rho_0}(1-\widehat{\chi}_\rho(1)) = 0.$$

If $\lambda \in \Lambda[0,\infty)$, then $\{I_\rho\}$ is an approximation process in $X^0_{2,\lambda}$ and there exists a constant C such that, for all $f \in X^0_{2,\lambda}$,

$$\|f - I_\rho(f)\|_{X^0_{2,\lambda}} \le C\theta_{2,\lambda}\left(f, \sqrt{1-\widehat{\chi}_\rho(1)}\right). \tag{9.11}$$

Remark 9.3.1 Theorem 9.3.2 generalizes and improves several results given for some articular kernels as the Gauss-Weierstrass integral ([4], [5]), de la Vallé Poussin-type means ([11]), Abel means ([10] and [8], see also [12]). In some of these papers the situation is different to the one considered here. They assume the operators act between two different Hölder spaces, where the first one is compactly embedded in the second. Say $\omega(t) = t^\alpha$ and $\eta(t) = t^\beta$, $\beta < \alpha$, and $I_\rho : X_{2,\alpha} \to X_{2,\beta}$. Of course these cases are covered by Theorem 9.3.2. The approach presented here is simpler and provides an unified presentation.

Remark 9.3.2 We restrict the presentation in Theorem 3 to the case of positive kernels, but similar results can be obtained for suitable linear combination of this operators, for instance, the so called m-singular integrals. That is, if m is a positive integer define (see [6])

$$I_{\rho,m}(f,x) = \frac{(-1)^{m+1}}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}f(x-ku)\right]\chi_\rho(u)du.$$

This approach is useful when we should work with higher order moduli of smoothness. For these operators there are analogous to Theorems 9.3.1 and 9.3.2.

9.4 Cesàro and Riesz means

For $f \in L^1_{2\pi}$ put $P_k f(x) = a_k(f)\cos(kx) + b_k(f)\sin(kx)$, $x \in \mathbb{R}$, where $a_k(f)$ and $b_k(f)$ are the Fourier coefficients of f. Let $s_n f = \sum_{k=0}^{n} P_k f$ and denote by $\sigma_k f$ the arithmetic means of $\{s_0 f, \dots s_n f\}$.

9.4.1 Cesàro means

For a real $r > 0$ let the numbers C_n^r be defined by

$$\sum_{n=0}^{\infty} C_n^r x^n = \frac{1}{(1-x)^{r+1}}, \quad |\, x \,| < 1.$$

The Cesàro means of order r, $C_{n,r}$ are defined by

$$C_{n,r} f(x) = \frac{1}{C_n^r} \sum_{k=0}^{n} C_{n-k}^r \sigma_k f(x). \tag{9.12}$$

In order to use the results of Section 2 we should present an auxiliary, but important proposition.

Proposition 9.4.1 *For each $r > 0$ and $1 \leq p \leq \infty$ there exist a positive constant C such that, for any $f \in L^p_{2\pi}$, $n \in \mathbb{N}$ and $t \in [0, 2\pi]$,*

$$\|f - C_{n,r} f\|_p \leq C\omega_2\left(f, \frac{1}{n}\right)_p$$

and

$$\omega(C_{n,r} f, t)_p \leq C\omega_2(f, t)_p.$$

Proof. The first inequality is known. For instance, it follows from the representation (see [9])

$$C_{n,r} f(x) = -\frac{r}{2\pi} \int_1^{\infty} \frac{\Delta^2_{t/(n+1)} f(x)}{t^2} dt + \tau_{n,r}(f, x), \tag{9.13}$$

where, for some positive constants $C_1 = C_1(p)$ and $C_2 = C_2(p)$,

$$C_1\omega_2\left(f, \frac{1}{n+1}\right)_p \leq \|\tau_{n,r}(f)\|_p \leq C_2\omega_2\left(f, \frac{1}{n+1}\right)_p.$$

From the Banach-Steinhauss principle one has the Lebesgue numbers

$$\int_0^{2\pi} \left| \frac{1}{C_n^r} \sum_{k=0}^{n} C_{n-k}^r D_k(t) \right| dt,$$

are uniformly bounded, where D_k is the Dirichlet's kernel. From this remark and (9.12) the second inequality follows.□

Theorem 9.4.1 *For each $r > 0$, $\lambda \in \Lambda[0, \infty)$ and $1 \leq p \leq \infty$ there exists a positive constant C such that, for any $f \in L_{2\pi}^p$ and $n \in \mathbb{N}$,*

$$\|f - C_{n,r}f\|_{\omega,\lambda} \leq C\theta_{\omega,\lambda}\left(f, \frac{1}{n}\right),$$

where $C_{n,r}$ is the Cesàro mean and $\theta_{\omega,\lambda}$ is defined by (9.2) with $\omega(f, h) = \omega_2(f, h)_p$.

Proof. It follows from Theorem 9.2.1 and Proposition 9.4.1.□

Remark 9.4.1 For $0 \leq \beta < \alpha \leq 1$, in [7] the Cesàro means where considered from the space $Lip_{1,\alpha}$ into $Lip_{1,\beta}$. The last theorem improves and extends the results in [7].

9.4.2 Riesz means

For $\xi \geq 1$, $\gamma > 0$ and $f \in L_{2\pi}^1$, the Riesz means are defined by

$$R_{n,\gamma,\xi}(f, x) = \sum_{k=0}^{n} \left(1 - \left\{\frac{k}{n+1}\right\}^\gamma\right)^\xi P_k f(x). \tag{9.14}$$

In this section, we derive estimations for the rate of convergence of Riesz means with the help of Cesàro means.

We need some ideas from [14]. We associate to a real sequence $\eta = \{\eta_k\}$ and $f \in X$ a formal series as follows

$$T_\eta(f) \sim \sum_{k=0}^{\infty} \eta_k P_k(f),$$

where $P_k f(x)$ is defined as above.

We say that the sequence $\eta = \{\eta_k\}$ is a bounded multiplier, if there exists a constant C such that, for each trigonometric polynomial P, $\|T_\eta P\| \leq C\|P\|$. In such a case we denote by $\|\eta\|$ the norm of the operator $T_\eta : \Pi \to \Pi$.

Let $BV_{\xi+1}$ be the space of all functions $m \in C_0[0,\infty)$ such that $m, \ldots, m^{(\xi-1)} \in AC_{loc}$, $m^{(\xi)} \in BV_{loc}$ and

$$\int_0^\infty t^\xi |dm^{(\xi)}(t)| < \infty.$$

In this space we consider the norm

$$\|m\|_{BV_{\xi+1}} = \int_0^\infty t^\xi |dm^{(\xi)}(t)| < \infty.$$

Let Φ be a non-negative, strictly increasing function such that $\lim_{t\to 0+} \Phi(t) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$, and let Φ possess $(\xi+1)$ continuous derivatives on $(0,\infty)$ with

$$|t^k \Phi^{(k+1)}(t)| \leq D\Phi'(t), \quad t > 0, \quad 0 \leq k \leq \xi.$$

Let $D : \Pi \to X$ be the linear operator defined by $D(A_k(x)) = \Phi(k)A_k(x)$, where Φ is fixed as above and $A_k(x) = \sin(kx)$ or $A_k(x) = \cos(kx)$.

Take $\lambda(s) = s^\alpha$. For $f \in X^0_{\omega,\lambda}$ and $t \geq 0$ define a K-functional by $K(f,t) = \inf_{g\in\Pi}(\|f-g\|_X + t\|Dg\|_X)$. We use this K-functional to estimate the rate of convergence of the Riesz means in $X^0_{\omega,\lambda}$.

Theorem 9.4.2 *If the Cesàro means of order $\delta > 0$, (C,δ), are uniformly bounded on $X^0_{\omega,\lambda}$, $\xi \geq \delta$ $(\xi \in \mathbb{N})$ and*

$$R_{n,\gamma,\xi} f = \sum_{k=0}^{n} \left(1 - \left\{\frac{k}{n+1}\right\}^\gamma\right)^\xi P_k f, \quad f \in X^0_{\omega,\lambda},$$

then there exist positive constants C_1 and C_2 such that, for any $f \in X^0_{\omega,\lambda}$ and $n \in \mathbb{N}$,

$$C_1 K\left(f, \frac{1}{(n+1)^\gamma}\right) \le \|f - R_{n,\gamma,\xi}(f)\|_{\omega,\lambda} \le C_2 K\left(f, \frac{1}{(n+1)^\gamma}\right). \quad (9.15)$$

Proof. The proof is similar to the one of theorem 1 in [13]. First observe that

$$R_{n,\gamma,\xi} f = \sum_{k=0}^{n} m\left(\left\{\frac{k}{n+1}\right\}^\gamma\right) P_k f,$$

where $m(t) = (1-t)_+^\xi \in BV_{\xi+1}$; here we use the notation $(1-t)_+ = 1-t$ whenever $t < 1$ and $= 0$ otherwise. It follows from theorem 3.9 in [14] that the sequence of operators $\{R_{n,\gamma,\xi}\}_n$, $R_{n,\gamma,\xi} : X^0_{\omega,\lambda} \to X^0_{\omega,\lambda}$, is uniformly bounded. Since $\|R_{n,\gamma,\xi} f - f\|_{\omega,\lambda} \to 0$, as $n \to \infty$ for $f \in \Pi$, the Banach-Steinhaus theorem implies that $\{R_{n,\gamma,\xi}\}_n$ is an approximation process on $X^0_{\omega,\lambda}$.

To obtain the first inequality in (9.15), notice that

$$K\left(f, \frac{1}{(n+1)^\gamma}\right) \le \|f - R_{n,\gamma,\xi} f\|_{\omega,\lambda} + \frac{1}{(n+1)^\gamma}\|D R_{n,\gamma,\xi} f\|_{\omega,\lambda}.$$

We need to estimate the second term. For $k \ge 1$,

$$\frac{1}{(n+1)^\gamma} P_k(D R_{n,\gamma,\xi} f) =$$

$$\frac{(k^\gamma/(n+1)^\gamma)(1-(k^\gamma/(n+1)^\gamma))_+^\xi}{1-(1-k^\gamma/(n+1)^\gamma)_+^\xi} P_k(f - R_{n,\gamma,\xi} f).$$

Notice that the function $m(t) = t(1-t)_+^\xi/(1-(1-t)_+^\xi) \in BV_{\xi+1}$ (see [14] pag. 2885). Also observe that the collection of multipliers $\{\eta(n)\}_{n\in\mathbb{N}}$ given by

$$\eta(n)_k = \begin{cases} \frac{(k^\gamma/(n+1)^\gamma)(1-(k^\gamma/(n+1)^\gamma))_+^\xi}{1-(1-k^\gamma/(n+1)^\gamma)_+^\xi} & 1 \le k \le n, \\ 0 & k > n. \end{cases}$$

define a family of linear operators $M_n : X^0_{\omega,\lambda} \longrightarrow \Pi$. So

$$\frac{1}{(n+1)^\gamma}(DR_{n,\gamma,\xi} f) = M_n(f - R_{n\gamma,\xi} f).$$

Using again theorem 3.9 of [14], we have that $\{M_n\}$ is a bounded family of linear operators. Hence $K(f,(n+1)^{-\gamma}) \leq C_3 \|f - R_{n,\gamma,\xi} f\|_{\omega,\lambda}$, for some positive constant C_3. The inequality is obtained by putting $C_1 = C_3^{-1}$

For the second inequality in (9.15) fix an arbitrary $g \in \Pi$. Notice that

$$\begin{aligned} \|f - R_{n,\gamma,\xi} f\|_{\omega,\lambda} &\leq \|(f-g)\|_{\omega,\lambda} + \|R_{n,\gamma,\xi}(f-g)\|_{\omega,\lambda} + \|g - R_{n,\gamma,\xi} g\|_{\omega,\lambda} \\ &\leq C\|f-g\|_{\omega,\lambda} + \|g - R_{n,\gamma,\xi} g\|_{\omega,\lambda}. \end{aligned}$$

We must to estimate the second term. First observe that

$$P_k(g - R_{n,\gamma,\xi} g) = \frac{1-(1-k^\gamma/(n+1)^\gamma)^\xi_+}{k^\gamma/(n+1)^\gamma} \frac{1}{(n+1)^\gamma} P_k(Dg).$$

It is known that $m(t) = (1-(1-t)^\xi_+)/t \in BV_{\xi+1}$. Here the multipliers are

$$\eta(n)_k = \begin{cases} \frac{1-(1-k^\gamma/(n+1)^\gamma)^\xi}{k^\gamma/(n+1)^\gamma} & 1 \leq k \leq n, \\ 0 & k > n. \end{cases}$$

These multipliers define a collection of bounded linear operators $\{T_n\}$ (3.9 of [14]). So

$$g - R_{n,\gamma,\xi} g = \frac{1}{(n+1)^\gamma} T_n(Dg), \qquad n \in \mathbb{N}.$$

Then there exists a positive constant C_4 independent of g such that

$$\|T_n(Dg)\|_{\omega,\lambda} \leq C_4 \|Dg\|_{\omega,\lambda}.$$

Taking $C_2 = \max\{C, C_4\}$, we have

$$\|f - R_{n,\gamma,\xi} f\|_{\omega,\lambda} \leq C_2 \|f-g\|_{\omega,\lambda} + \frac{C_2}{(n+1)^\gamma} \|(Dg)\|_{\omega,\lambda}.$$

Therefore, because g was chosen arbitrary, we obtain

$$\|f - R_{n,\gamma,\xi} f\|_{\omega,\lambda} \leq C_2 K\left(f, \frac{1}{(n+1)^\gamma}\right). \square$$

9.5 References

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